b) Der Formalismus der linearen Antwort I
(Methode der Greenschen Fkt.)

- dynam. Eigenschaften eines Systems?

äußerer Stimulus/erregende Kraft $\longrightarrow$ Reaktion/Antwort

Bsp: $A_\omega = \chi(\omega)\frac{F_\omega}{m}$ ... „linear"

- Ziel: „allgemeinste lineare Antwort"

- Bsp: harm. Oszillator: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \hat{F}(t) = \frac{F(t)}{m}$ (5.40)

beliebige Kraft $\hat{F}(t)$ $\xrightarrow[\text{Antwort}]{\text{„allg. lineare}}$ $x_{part}(t) = \underbrace{\int_{-\infty}^{t} G(t-t')\hat{F}(t')dt'}_{\text{„Faltung"}}$ (5.41)

(bzw: $I(t) = \int_{-\infty}^{t}\hat{F}(t')dt'$

... Kraftstoß)

(i) $\int_{-\infty}^{t}\dots$ Kausalitätsprinzip
(„Wirkung nach der Ursache")

(ii) $G(t-t')$ ... „Greensche Fkt"

- Führe ein: $G_c(t-t') = \begin{cases} G(t-t'), & t>t' \\ 0, & t<t' \end{cases}$ ... kausale Greensche Fkt.

(5.41) $\longrightarrow$ $\boxed{x_{part}(t) = \int_{-\infty}^{\infty} G_c(t-t')\hat{F}(t')dt'}$ (5.42)

- Bestimmungsgleichung für $G_c(t-t')$:

(5.42) in (5.40) $\longrightarrow$ $$\hat{F}(t) = \int_{-\infty}^{\infty} \underbrace{\left[\frac{d^2}{dt^2} + 2\gamma_0 \frac{d}{dt} + \omega_0^2\right]}_{\hat{L}(t)} G_c(t-t')\,\hat{F}(t')\,dt' \quad (5.43)$$

$\hat{L}(t)$ Differentialoperator

$\longrightarrow$ formal: $\underbrace{\hat{L}(t) G_c}_{= \text{"}1\text{"}} \hat{F} = \hat{F}$

= "1" ... Eins-Operator

hier „Zeitraum": Führe ein „δ-Funktion" (Distribution)

sodaß: $$\boxed{\hat{F}(t) = \int_{-\infty}^{\infty} \underbrace{\delta(t-t')}_{} \hat{F}(t')\,dt'} \quad (5.44)$$

1-Operator im Zeitraum: $\delta(t-t')$ stark gepeakt bei $t=t'$

vgl. mit (5.43) $\longrightarrow$ $$\boxed{\hat{L}(t)\, G_c(t-t') = \delta(t-t')} \quad (5.45)$$

c) Die Diracsche „δ-Funktion"

- Grundeigenschaft: $$\boxed{\int_{-\infty}^{\infty} \delta(x-x') f(x')\,dx' = f(x)} \quad (5.46)$$

$\delta(x-x')$ ... 1-Operator im Raum der Fktn. $f(x)$

[vgl. $a_i = \sum_j \delta_{ij} a_j$, $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$

hier: $\sum_j \delta_{ij} \longrightarrow \int \delta(x-x') \dots dx'$]

• „δ-Fkt." als Grenzfall <u>stetiger</u> Fktn.:

<u>Satz</u>:

$$\boxed{\begin{aligned} &\delta(x-x') = \text{"}\lim_{\varepsilon \to 0} \delta_\varepsilon(x-x')\text{"} \\ &\text{wobei } 1.\ \delta_\varepsilon(x-x') \underset{\xi = \frac{x-x'}{\varepsilon}}{=\!=} \frac{1}{\varepsilon} \underbrace{g(\xi)}_{\text{"gutartig"}} \\ &2.\ \int_{-\infty}^{\infty} \delta_\varepsilon(x-x') \underbrace{d(x-x')}_{\varepsilon\, d\xi} = \int_{-\infty}^{\infty} g(\xi)\, d\xi = 1 \\ &3.\ \text{"}\dots\text{"}: \text{ in (S.46): erst } \int \dots, \text{ dann } \lim_{\varepsilon \to 0} \end{aligned}} \quad \text{(S.47)}$$

(S.47) → (S.46)? <u>Bew</u>:

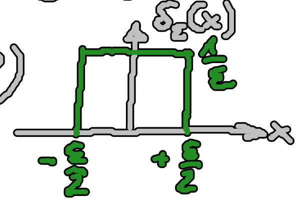
$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \underbrace{\delta_\varepsilon(x-x')}_{\frac{1}{\varepsilon} g(\xi)} f(x')\, dx' \underset{\substack{\frac{x-x'}{\varepsilon} = \xi \\ \varepsilon\, d\xi = -dx'}}{=\!=} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \underbrace{g(\xi) f(x - \varepsilon\xi)}_{\text{"ordentlich"}} d\xi$$

$$\longrightarrow \int \lim_{\varepsilon \to 0} \dots$$

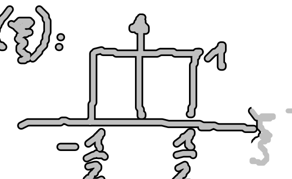
$$= f(x) \underbrace{\int_{-\infty}^{\infty} g(\xi)\, d\xi}_{1} = f(x)$$

→ i.f. Rechne mit $\delta(x-x')$ ohne $\lim_{\varepsilon \to 0}$

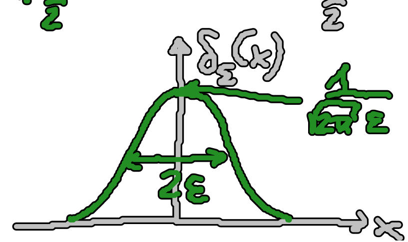
• Bsp: (i) $\delta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} & \text{für } |x| \leq \frac{\varepsilon}{2} \\ 0 & \text{"} \quad |x| > \frac{\varepsilon}{2} \end{cases}$ (S.48)



[g(ξ):

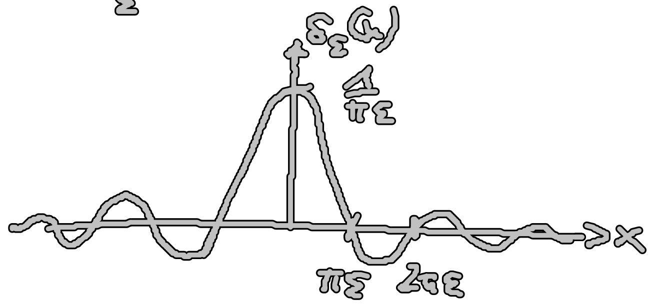


(ii) $\delta_\varepsilon(x) = \frac{1}{\sqrt{2\pi}\,\varepsilon} e^{-x^2/2\varepsilon^2}$ (S.49) … Gauß-Fkt. (Glockenfkt.)



(iii) $\delta_\varepsilon(x) = \frac{1}{\pi\varepsilon}\frac{1}{1+\left(\frac{x}{\varepsilon}\right)^2}$ ... Lorentz-Kurve (S.50)

(iv) $\delta_\varepsilon(x) = \frac{1}{\pi\varepsilon}\frac{\sin x/\varepsilon}{x/\varepsilon} \overset{\text{s.Ü.}}{=} \frac{1}{2\pi}\int_{-\frac{1}{\varepsilon}}^{1/\varepsilon} dq\, e^{iqx}$ (S.51) ... Spaltbeugungs-fkt. der Optik



· <u>Eigenschaften der δ-Funktion</u>

(i) $\delta(x) = \begin{cases} 0, & x \neq 0 \\ \text{„}\infty\text{"}, & x = 0 \end{cases}$ (S.52)

(ii) $\int_a^b \delta(x-x')\, f(x')\, dx' = \begin{cases} f(x), & a < x < b \\ 0, & \text{sonst} \end{cases}$ (S.53)

(iii) $\delta(ax) = \frac{1}{|a|}\delta(x)$ (S.54)

$\delta(-x) = \delta(x)$ ... gerade Fkt (S.55)

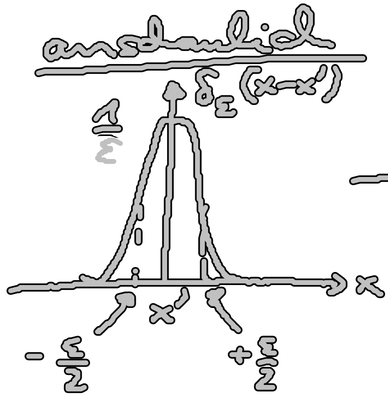
(iv) $\delta(x) \overset{(S.51)}{=} \int_{-\infty}^{\infty} e^{\pm iqx} \frac{dq}{2\pi}$ (S.56)

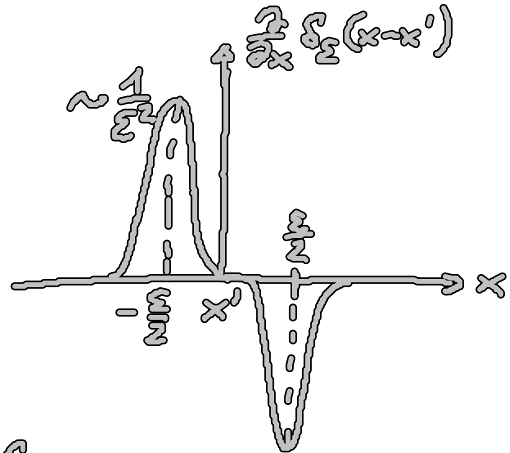
... Fourier-Darstellung

(v) $\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\delta(x-x')\, f(x')\, dx' = -\int \left[\frac{\partial}{\partial x'}\delta(x-x')\right] f(x')\, dx'$

$= f'(x)$ (S.57)

anschaulich





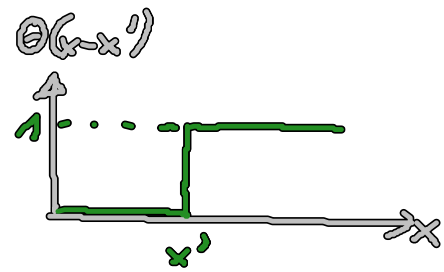
$$\int \frac{\partial}{\partial x} \delta_\varepsilon(x-x') f(x')\, dx' \sim \frac{1}{\varepsilon}\int \left[\delta_\varepsilon\left(x-x'+\frac{\varepsilon}{2}\right) - \delta_\varepsilon\left(x-x'-\frac{\varepsilon}{2}\right)\right] f(x')\, dx'$$

$$\approx \underbrace{\frac{1}{\varepsilon}\left[f\left(x+\frac{\varepsilon}{2}\right) - f\left(x-\frac{\varepsilon}{2}\right)\right]}_{\text{zentrierte Abl.}} \longrightarrow f'(x),\quad \varepsilon \to 0$$

Beweis: (iii)/(v) : Übungen

• **Stufenfkt.**

(i) $$\boxed{\Theta(x-x') = \begin{cases} 1, & x-x'>0 \\ 0, & x-x'<0 \end{cases}} \quad (5.58)$$



(ii) $$\Theta(x) \underset{(5.53)}{=} \int_{-\infty}^{x} \delta(x')\, dx' \quad (5.59)$$

$$\longrightarrow \boxed{\delta(x) = \frac{d}{dx}\Theta(x)} \quad (5.60)$$

d) Formalismus der linearen Antwort II

• Es galt: $\mathcal{L}(t)\, G_c(t-t') = \underbrace{\delta(t-t')}_{\delta\text{-Kraft}}$, $\mathcal{L}(t) = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2$ (5.61)

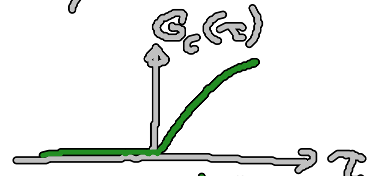
, $\int \delta(t-t')\, dt = 1$ … δ-Kraftstoß

• Bestimmung der Greensche Fkt. aus Stetigkeits- und Sprungbed.:

(i) Kausalität: $\boxed{G_c(\tau) = 0,\ \tau = t-t' < 0}$

(ii) Stetigkeit: $G_c(\varepsilon) - \underbrace{G_c(-\varepsilon)}_{=0} = 0, \quad \varepsilon \to 0$

$$\longrightarrow \boxed{G_c(0) = 0}$$



Knick [s. (iii)]