c) Drehimpuls

$$\boxed{\underline{L}_I = \underline{L}_S + \underline{L}} \qquad (10.18)$$

$$\underline{L} = \sum_\nu m_\nu \underline{r}_\nu \times (\underline{\omega} \times \underline{r}_\nu) \qquad (10.20)$$

$$= \underline{\underline{\Theta}}\,\underline{\omega} \qquad (10.24)$$

$$\text{mit } \underline{\underline{\Theta}} = \sum_\nu m_\nu \left[|\underline{r}_\nu|^2 \underline{\underline{1}} - \underline{r}_\nu(t) \otimes \underline{r}_\nu(t) \right]$$

wobei $(\underline{r}_\nu \otimes \underline{r}_\nu)\underline{\omega} = \underline{r}_\nu\, \underline{r}_\nu \cdot \underline{\omega}$

- Komponentendarstellung bzgl. ONB $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$\underline{e}_i \cdot \big|\ \underline{L} = \underline{\underline{\Theta}}\,\underbrace{\underline{\omega}}_{\omega_j \underline{e}_j} \longrightarrow L_i = \underline{e}_i \cdot \underline{\underline{\Theta}}\,\underline{e}_j\, \omega_j$

$$\longrightarrow \boxed{L_i = \Theta_{ij}\,\omega_j \quad \text{mit } \Theta_{ij} = \underline{e}_i \cdot \underline{\underline{\Theta}}\,\underline{e}_j = \sum_\nu m_\nu \left[|\underline{r}_\nu|^2 \delta_{ij} - x_{\nu i}\, x_{\nu j} \right]} \qquad (10.25)$$

$x_{\nu j} = \underline{r}_\nu \cdot \underline{e}_j$!!

Matrix $\underline{\underline{\Theta}}$ mit $[\Theta]_{ij} = \Theta_{ij}$

- symmetr. Tensor 2St: $\Theta_{ij} = \Theta_{ji} \longrightarrow$ 6 unabh. Komp.

$$\underline{\underline{\Theta}} = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{12} & \Theta_{22} & \Theta_{23} \\ \Theta_{13} & \Theta_{23} & \Theta_{33} \end{pmatrix}$$

Hauptdiagonale: $\Theta_{11}, \Theta_{22}, \Theta_{33}$ ... Trägheitsmomente

Nebendiagonalelemente: $\Theta_{12}, \Theta_{13}, \Theta_{23}$ ... Deviationsmomente

$\longrightarrow$ Lagerkräfte Bsp. Rad

- Trägheitsmoment bzgl. Achse $\hat{\underline{\nu}}$ mit $|\hat{\underline{\nu}}| = 1$: $\Theta_{\nu\nu} = \hat{\underline{\nu}} \cdot \underline{\underline{\Theta}}\,\hat{\underline{\nu}}$

Bsp: $\Theta_{11} = \underline{e}_1 \cdot \underline{\underline{\Theta}}\,\underline{e}_1$

- $\underline{\underline{\Theta}} = \underline{\underline{\Theta}}(t)$ (vgl. 10.24): Wo steckt die Zeitabh. rel. zum IS?

(i) ONB des KS: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ ... körperfest!

$$\underline{\underline{\Theta}}(t) = \Theta_{ij}\, \underline{e}_i(t) \otimes \underline{e}_j(t) \,, \quad \Theta_{ij} = \underline{e}_i(t) \cdot \underline{\underline{\Theta}}\, \underline{e}_j(t) \qquad (10.27)$$

... zeitunabh.

(ii) ONB des IS: $\{\underline{e}_{I1}, \underline{e}_{I2}, \underline{e}_{I3}\}$ ... raumfest!

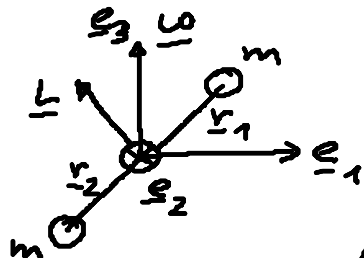
$$\underline{\underline{\Theta}}(t) = \Theta_{Iij}(t)\, \underline{e}_{Ii} \otimes \underline{e}_{Ij} \,, \quad \Theta_{Iij}(t) = \underline{e}_{Ii} \cdot \underline{\underline{\Theta}}(t)\, \underline{e}_{Ij} \qquad (10.28)$$

... zeitabh.

• kontinuierl. Massenverteilung:

$$\underline{\underline{\Theta}} \underset{(10.7)}{=} \overbrace{\int d^3r}^{\leftarrow \sum_\nu m_\nu \dots}\, \rho(\underline{r})\, \left[|\underline{r}|^2 \underline{\underline{1}} - \underline{r} \otimes \underline{r}\right] \qquad (10.29)$$

• Bsp: „Hantel"



$$\underline{r}_1 = \frac{d}{\sqrt{2}}(\underline{e}_1 + \underline{e}_3) \qquad |\underline{r}_i| = d$$

$$\underline{r}_2 = -\frac{d}{\sqrt{2}}(\underline{e}_1 + \underline{e}_3)$$

$$\Theta_{11} = \Theta_{33} = md^2\, 2\left(1 - \tfrac{1}{2}\right) = md^2$$

$$\sum_\nu m_\nu \left[|\underline{r}_\nu|^2 \delta_{ij} - x_{\nu i} x_{\nu j}\right] \qquad \Theta_{22} = 2md^2$$

$$\Theta_{13} = \Theta_{31} = md^2\, 2\left(-\tfrac{1}{2}\right) = -md^2$$

$$\Theta_{ij} = 0 \,, \text{ sonst}$$

$$\longrightarrow \underline{\underline{\Theta}} = md^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad (10.30)$$

$$\underline{\omega} = \omega \underbrace{\underline{e}_3}_{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} : \quad \underline{L} = \underline{\underline{\Theta}}\, \underline{\omega} = md^2 \omega \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

e) kinet. Energie:

$$T = \frac{1}{2} \sum_\nu m_\nu\, \dot{\underline{r}}_{I\nu}^2$$

$$\text{mit } \dot{\underline{r}}_{I\nu} = \dot{\underline{R}} + \underline{\omega} \times \underline{r}_\nu$$

(i) $\underline{R} = \underline{R}_S$

$$\xrightarrow{\sum_\nu m_\nu \underline{r}_\nu = 0} \boxed{\begin{aligned} T &= T_{trans} + T_{rot} \\ &= \frac{1}{2} M \dot{\underline{R}}_S^2 + \frac{1}{2} \sum_\nu m_\nu (\underline{\omega} \times \underline{r}_\nu)^2 \end{aligned}} \qquad (10.31)$$

Untersuche: $\underline{\omega}\cdot\underline{L} \underset{(10.20)}{=} \sum_{\nu} m_\nu \underbrace{\underline{\omega}\cdot\left[\underline{r}_\nu\times(\underline{\omega}\times\underline{r}_\nu)\right]}_{\underline{a}\cdot(\underline{b}\times\underline{c}) \overset{!}{=} \underline{c}\cdot(\underline{a}\times\underline{b})}$

$\underline{a}\cdot(\underline{b}\times\underline{c})$
$= \varepsilon_{ijk}\, a_i\, b_j\, c_k$

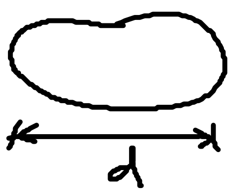
$$= \sum_{\nu} m_\nu \left(\underline{\omega}\times\underline{r}_\nu\right)^2$$

$$\longrightarrow \boxed{T_{rot} = \frac{1}{2}\,\underline{\omega}\cdot\underline{L} = \frac{1}{2}\,\underline{\omega}\cdot\underline{\underline{\Theta}}\,\underline{\omega} = \frac{1}{2}\,\omega_i\,\Theta_{ij}\,\omega_j} \quad (10.32)$$

... für Rotation um Aufpunkt

(ii) $\dot{\underline{R}} = 0$: $\boxed{T = T_{rot}}$ (10.33)

f) <u>potentielle Energie:</u> $U(\underline{r}_{I\nu})$ für m_ν, $\nu = 1,\dots,N$



$\underline{r}_{I\nu} = \underline{R} + \underline{r}_\nu$ mit $|\underline{r}_\nu| \leq d$

$$U(\underline{r}_{I\nu}) \overset{\text{Taylor}}{=} U(\underline{R}) + \underline{r}_\nu\cdot\operatorname{grad}_\nu U(\underline{r}_{I\nu})\Big|_{\underline{R}}$$

falls: $d\,|\operatorname{grad}_\nu U(\underline{r}_{I\nu})| \ll U(\underline{R})$

$$\longrightarrow \boxed{U(\underline{r}_{I\nu}) \approx U(\underline{R})} \quad (10.34)$$

g) <u>Eigenschaften von $\underline{\underline{\Theta}}$:</u>

(1) <u>Hauptachsentrafo:</u> (10.35)

· Eigenwertgl.:

$$\boxed{\begin{array}{l} \underline{\underline{\Theta}}\,\underline{e}^{(i)} = \Theta_i\,\underline{e}^{(i)} \\ \left.\begin{array}{l}\Theta_i \geq 0 \;\dots\; \text{Hauptträgheitsmomente} \\ \{\underline{e}^{(1)}, \underline{e}^{(2)}, \underline{e}^{(3)}\} \;\dots\; \text{Hauptachsen-system}\end{array}\right\} \text{von } \underline{\underline{\Theta}} \end{array}}$$

$$\boxed{\text{mit } \underline{e}^{(i)}\cdot\underline{e}^{(j)} = \delta_{ij}}$$

· Diagonalgestalt von $\underline{\underline{\Theta}}$: (10.36)

$$\tilde{\Theta}_{ij} = \underline{e}^{(i)}\cdot \underbrace{\underline{\underline{\Theta}}\,\underline{e}^{(j)}}_{\Theta_j\, \underline{e}^{(j)}} = \underbrace{\Theta_j\, \delta_{ij}}_{\text{keine Summenkonvention!}} \longrightarrow \tilde{\underline{\underline{\Theta}}} = \begin{pmatrix}\Theta_1 & & 0\\ & \Theta_2 & \\ 0 & & \Theta_3\end{pmatrix}$$

also: $$\boxed{\underline{\underline{\Theta}} = \sum_{i=1}^{3} \underbrace{\Theta_i}_{3\,\text{EW}}\; \underbrace{\underline{e}^{(i)} \otimes \underline{e}^{(i)}}_{3\,\text{Eulerwinkel}}} \quad (10.37)$$

3 EW 3 Eulerwinkel $\longrightarrow$ 6 unabh. Komp von $\underline{\underline{\Theta}}$

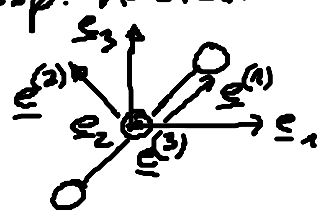
· Fälle: (i) $\Theta_1 \neq \Theta_2 \neq \Theta_3$: unsymmetr. Kreisel

(ii) $\Theta_1 = \Theta_2 \neq \Theta_3$: achsen " "

(iii) $\Theta_1 = \Theta_2 = \Theta_3$: Würfel oder Kugel! (oder Tetraeder)

· physikal: $\underline{\omega} \parallel \underline{e}^{(i)} \longrightarrow \underline{L} \parallel \underline{\omega}$ ... „stabile" Drehrichtungen!

· Bsp: Hantel:



$$\underline{\underline{\Theta}} = md^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}(\underline{e}_1 + \underline{e}_3) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \Theta_1 = 0$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}(-\underline{e}_1 + \underline{e}_3) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad \Theta_2 = 2md^2$$

$$\underline{e}^{(3)} = \underline{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \Theta_3 = 2md^2$$

$$\tilde{\underline{\underline{\Theta}}} = 2md^2 \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

(2) Trägheitsellipsoid: Fläche konst. Rot.energie

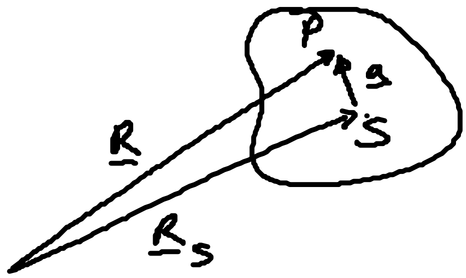
· $T = \frac{1}{2}\, \underline{\omega} \cdot \underline{\underline{\Theta}}\, \underline{\omega}$ mit $\underline{\rho} = \frac{\underline{\omega}}{\sqrt{T}}$ (10.38)

$$\longrightarrow \boxed{1 = \rho_i\, \Theta_{ij}\, \rho_j \xrightarrow[\text{achsentrafo}]{\text{Haupt-}} 1 = \Theta_1 \tilde{\rho}_1^2 + \Theta_2 \tilde{\rho}_2^2 + \Theta_3 \tilde{\rho}_3^2}$$

... Trägheitsellipsoid mit Halbachse $\frac{1}{\sqrt{\Theta_i}}$ (Flächen 2. Grades)

· Poinsotsche Konstruktion!

(3) Satz von Steiner



$\underline{\underline{\Theta}}^{(S)}$ bezogen auf S bekannt

$$\longrightarrow \quad \boxed{\underline{\underline{\Theta}}^{(P)} = \underbrace{M}_{\text{Gesamt-masse}}\left(|\underline{a}|^2 \mathbb{1} - \underline{a}\otimes\underline{a}\right) + \underline{\underline{\Theta}}^{(S)}}$$

... Trägheitstensor bzgl. Punkt P

Beweis: Übungen

10.3 Dynamik des starren Körpers (II): Eulersche Gln.

a) Dynam. Grundgl.

- Impulssatz:

(10.13)
(8.8) $\boxed{\dot{\underline{P}} = \sum_\nu \underline{F}_\nu^{(a)} = \underline{F}^{(a)}}$

(i) $\underline{R} = \underline{R}_S$: $\underline{P} = M\dot{\underline{R}}_S$ (10.16)

(ii) $\dot{\underline{R}} = 0$: $\underline{P} = \underline{\omega} \times M \underline{r}_S$ (10.17)

- Drehimpulssatz:

(10.14)
(8.14) $\dot{\underline{L}}_I = \underline{D}_I^{(a)} = \sum_\nu \underline{r}_{I\nu} \times \underline{F}_\nu^{(a)} = \underline{R} \times \underline{F}^{(a)} + \underline{D}$ (10.41)

$\underline{r}_{I\nu} = \underline{R} + \underline{r}_\nu$

$\boxed{\underline{D} = \sum_\nu \underline{r}_\nu \times \underline{F}_\nu^{(a)}}$

... Drehmoment bzgl. Aufpkt $\underline{R}$.

$$\overset{(10.18)}{\underset{(10.21)}{=}} \dot{\underline{L}}_S + \dot{\underline{L}}$$

(10.42)

(i) $\underline{R} = \underline{R}_S$: $\dot{\underline{L}}_S = (\underline{R}_S \times \underbrace{M\dot{\underline{R}}_S}_{\underline{P}})^{\cdot} \underset{\dot{\underline{R}}_S \times \dot{\underline{R}}_S = 0}{=} \underline{R}_S \times \dot{\underline{P}} = \underline{R}_S \times \underline{F}^{(a)}$

(ii) $\dot{\underline{R}} = 0$: $\dot{\underline{L}}_S = \underline{R} \times \dot{\underline{P}} = \underline{R} \times \underline{F}^{(a)}$ (10.43)

$$\Longrightarrow \quad \boxed{\dot{\underline{L}} = \underline{D}} \quad (10.44)$$

... Drehimpulssatz für starre Körper bzgl. Aufpkt $\underline{R}$ $\longrightarrow$ Rotationsbewegg

[vgl.: Newton: $\dot{\underline{p}} = \underline{F}$ mit $\underline{p} = m\underline{v}$ $\longleftrightarrow$ $\underline{L} = \underline{\Theta}\,\underline{\omega}$

$\underline{F}$ $\longleftrightarrow$ $\underline{D}$]

b) <u>Eulerschen Gln.</u>: