12.5. Lagrangesche Gln. 2. Art

- generalisierte Kräfte:

$$\boxed{Q_j = \sum_i \underline{F}_i^{(A)} \cdot \frac{\partial \underline{r}_i}{\partial q_j}} \quad (12.25)$$

$$\longrightarrow \quad \sum_i \underline{F}_i^{(A)} \cdot \delta \underline{r}_i = \sum_j Q_j \, \delta q_j$$

- Lagrangesche Gln. 2. Art (im allgemeinen)

$$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \quad (12.26)$$

T ... kinetische Energie

a) konservative Systeme: $\quad \underline{F}_i^{(A)} = -\underline{\nabla}_i U \longrightarrow \boxed{Q_j = -\frac{\partial U}{\partial q_j}}$ (12.28)

$$\longrightarrow \quad \boxed{\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \\ &L = T - U \text{ ... Lagrange-Fkt.} \end{aligned}}$$

(12.29) .... Lagrangesche Gln.

(12.30)

b) generalisiertes / geschwindigkeitsabh. Potential W:

- Annahme: $\boxed{\begin{aligned} &W = W(\{q_j\}, \{\dot{q}_j\}, t) \\ \text{mit } &Q_j = -\frac{\partial W}{\partial q_j} + \frac{d}{dt}\frac{\partial W}{\partial \dot{q}_j} \end{aligned}}$ existiert (12.31)

$$(12.26) \longrightarrow \boxed{\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \\ &L = T - W \end{aligned}} \quad (12.32)$$

• **Bsp:** geladenes Teilchen im elektromagnetischen Feld

– Lorentzkraft: $\underline{F} = q\,(\underline{E} + \underline{v} \times \underline{B})$ (12.33)

Ladung, elektr. Feld, Geschw. $\dot{\underline{r}}$, Magnetfeld

– o.B.

$$\boxed{\begin{aligned} &\underline{B} = \underline{\nabla} \times \underline{A} \\ &\underline{E} = -\underline{\nabla}\varphi - \frac{\partial}{\partial t}\underline{A} \\ &\varphi(\underline{r},t) \ldots \text{ skalares Potential} \\ &\underline{A}(\underline{r},t) \ldots \text{ Vektorpotential} \end{aligned}}$$ (12.34)

$\xrightarrow[\text{in (12.33)}]{(12.34)}$ $\underline{F} = q\left[-\underline{\nabla}\varphi - \frac{\partial}{\partial t}\underline{A} + \underline{v} \times (\underline{\nabla} \times \underline{A})\right]$

$\underline{v} = \underline{v}(t) \rightarrow \frac{\partial \underline{v}}{\partial x_i} = 0$

(i) $\underline{v} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{v} \cdot \underline{A}) - (\underline{v} \cdot \underline{\nabla})\underline{A}$

$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})$

(ii) $\frac{\partial}{\partial t}\underline{A} + (\underline{v} \cdot \underline{\nabla})\underline{A} = \frac{\partial}{\partial t}\underline{A} + \left(\dot{x}_1 \frac{\partial}{\partial x_1} + \dot{x}_2 \frac{\partial}{\partial x_2} + \dot{x}_3 \frac{\partial}{\partial x_3}\right)\underline{A}$

$= \frac{d}{dt}\underline{A}$

konvektive Ableitung

$\longrightarrow$ $\underline{F} = q\left[-\underline{\nabla}\varphi - \frac{d}{dt}\underline{A} + \underline{\nabla}(\underline{v} \cdot \underline{A})\right]$

$\longrightarrow$ $F_i = q\left[-\frac{\partial}{\partial x_i}(\varphi - \underline{v} \cdot \underline{A}) - \frac{d}{dt}\frac{\partial}{\partial v_i}\big(\underbrace{\underbrace{\underline{A} \cdot \underline{v}}_{A_j v_j} - \varphi}_{A_i}\big)\right]$

$\frac{\partial \varphi}{\partial v_i} = 0\,!$

$\xrightarrow{v_i = \dot{x}_i}$

$$\boxed{\begin{aligned} &F_i = -\frac{\partial W}{\partial x_i} + \frac{d}{dt}\frac{\partial W}{\partial \dot{x}_i} \\ &\text{mit } W = q\,(\varphi - \underline{A} \cdot \dot{\underline{r}}) \end{aligned}}$$ (12.35)

also: $\boxed{L = T - q\varphi + q\,\underline{A} \cdot \dot{\underline{r}}}$ (12.36)

c) Reibungskräfte

- Aufteilung der Kräfte: $\underline{F}_i^{(t)} =$ kons. Kraft + Rest
  $\hookrightarrow L = T - U$

$$\longrightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = Q_j} \quad (12.37)$$

- Reibungskraft:
  linearer Ansatz: $F_{i\alpha}^{(R)} = -\gamma_{\alpha\beta}^{(i)} \dot x_{i\beta}$ (12.38) $\quad \alpha, \beta = 1,2,3$
  (Reibungstensor: $\gamma_{\alpha\beta}^{(i)}$)

$$[\underline{F}_i^{(R)} = -\underline{\underline{\gamma}}^{(i)} \dot{\underline{x}}_i]$$

Bsp: Stokessche Reibung einer Kugel: $\gamma_{\alpha\beta}^{(i)} = \gamma\,\delta_{\alpha\beta}$

$$\longrightarrow F_{i\alpha}^{(R)} = -\gamma \dot x_{i\alpha}$$

$$\underline{F}_i^{(R)} \parallel \dot{\underline{r}}_i$$

- Rayleighsche Dissipationsfkt:

$$\boxed{W = \frac{1}{2}\sum_i \gamma_{\alpha\beta}^{(i)} \dot x_{i\alpha} \dot x_{i\beta}, \quad \gamma_{\alpha\beta}^{(i)} = \gamma_{\beta\alpha}^{(i)}} \quad (12.39)$$

$$\longrightarrow \boxed{F_{i\alpha}^{(R)} = -\frac{\partial W}{\partial \dot x_{i\alpha}} = -\gamma_{\alpha\beta}^{(i)} \dot x_{i\beta}} \quad (12.40)$$

Bsp: $\gamma_{\alpha\beta}^{(i)} = \gamma\,\delta_{\alpha\beta}$ $\quad W = \frac{1}{2}\sum_i \gamma \dot x_{i\alpha}^2 \quad F_{i\alpha}^{(R)} = -\frac{\partial W}{\partial \dot x_{i\alpha}} = -\gamma \dot x_{i\alpha}$ ✓

- physikal. Interpretation:

$$-\sum_i \underline{F}_i^{(R)} \cdot \dot{\underline{r}}_i \overset{(12.40)}{=} \sum_i \dot x_{i\alpha} \gamma_{\alpha\beta}^{(i)} \dot x_{i\beta} \overset{(12.39)}{=} 2W$$

$2W$ ... die von den Reibungskräften dissipierte Leistung

- generalisierte Kräfte:

$$Q_j \overset{(12.25)}{=} \sum_i \underline{F}_i^{(R)} \cdot \frac{\partial \underline{r}_i}{\partial q_j} \overset{(12.40)}{=} -\sum_i \frac{\partial W}{\partial \dot x_{i\alpha}} \underbrace{\frac{\partial x_{i\alpha}}{\partial q_j}}$$

Einsteinsche Summenkonv. für α, β

$$\dot x_{i\alpha} = \sum_j \frac{\partial x_{i\alpha}}{\partial q_j} \dot q_j + \frac{\partial}{\partial t} x_{i\alpha}$$

$$= \frac{\partial \dot x_{i\alpha}}{\partial \dot q_j} \quad [\text{vgl. } (12.22)]$$

$$\longrightarrow \boxed{Q_j = -\frac{\partial W}{\partial \dot q_j}} \quad (12.42)$$

$\dot q_j$: generalisierte Geschw.

Lagrangesche Gln:

$$(12.37) \longrightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} + \underbrace{\frac{\partial W}{\partial \dot q_j}}_{\text{„Reibung"}} = 0} \quad (12.43)$$

13. Hamiltonsche Prinzip

- bisher: d'Alembertsches Prinzip: $\sum_i \left(F_i^{(t)} - m_i \ddot{\underline{r}}_i\right)\cdot \delta \underline{r}_i = 0$

  $\longrightarrow$ Differentialprinzip: $\delta \underline{r}_i$ zur Zeit t

- jetzt: Integralprinzip: virtuelle Verrückung der gesamten Bahn & Extremal "

  $\underbrace{\qquad\qquad\qquad\qquad}$

  Mechanik, Elektromagnetismus, komplexe Materialien, Rel. Theorie, fundamentale Ww der Natur

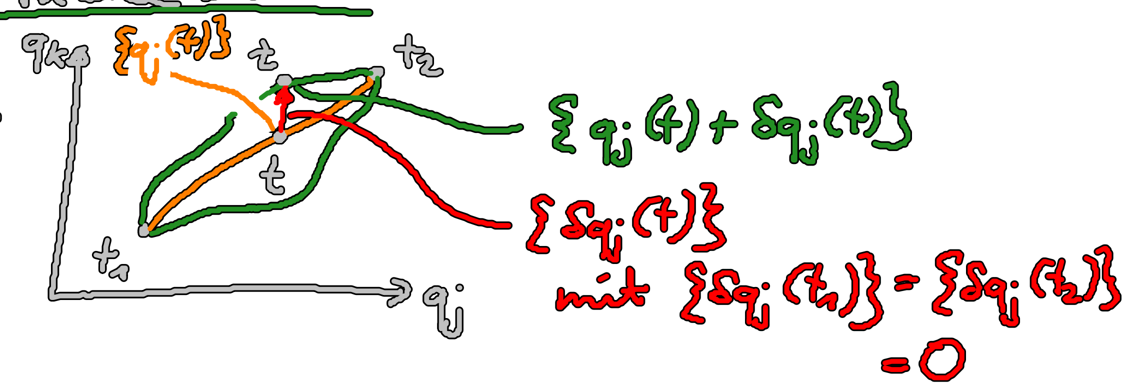
13.1. Hamiltonsches Prinzip („der kleinsten Wirkung") [historisch]

- mechan. System: s generalisierte, voneinander unabh. Koord.

  $q_1(t) \dots q_s(t) = \{q_j(t)\}$ (13.1)

- physikal / virtuelle Bahnen:

2D-Schnitt des Konf. raumes



• **Hamiltonsches Prinzip:**

$$\boxed{\begin{array}{l}\text{Die physikal. Bahn eines Systems zwischen } \{q_j(t_1)\} \\ \text{und } \{q_j(t_2)\} \text{ im Konfigurationsraum ist durch} \\ \text{ein Extremum der Wirkung } S = \int_{t_1}^{t_2} L(\{q_j\},\{\dot q_j\},t)\,dt \\ \text{ausgezeichnet:} \\ \qquad \underbrace{\delta S}_{\text{„Variation von } S\text{"}} = 0 \qquad\qquad (13.2) \\ L \text{ heißt Lagrange-Fktn.}\end{array}}$$

Bem.: (i) δS = Änderung von S bei

$\{q_j(t)\} \longrightarrow \{q_j(t) + \delta q_j(t)\}$

(ii) Extremum: $\delta S = 0$

Minimum oder Maximum!

(iii) Einheit von S: Energie × Zeit

• konservative Kräfte/generalisiertes Potential U:

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot q_j} \qquad (12.31)$$

Beh.:

$$\boxed{\begin{array}{l} L = T - U \\ \delta S = 0 \longrightarrow \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot q_j} - \dfrac{\partial L}{\partial q_j} = 0 \end{array}} \qquad (13.3)$$

... Lagrangesche Gln. (2. Art)

Beweis: Kap 13.3

• fundamentale Ww in der Natur

1. Gravitation
2. EM Ww
3. schwache Ww („Neutrinos")
4. Starke Ww (Kernkräfte)

} herleitbar über Hamiltonsches Prinzip & L