6.2.08

Klausur :

- für eine Teilnahme muß jeder einmal im Tutorium vorgerechnet haben
- keine Hilfsmittel, Papier wird gestellt.
- bitte Ausweis mit Bild mitbringen.

Freitag diese Woche <u>keine</u> Vorlesung mehr!

6.2.2008

1 dimensionale Streutheorie

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) = E\Psi(x)$$

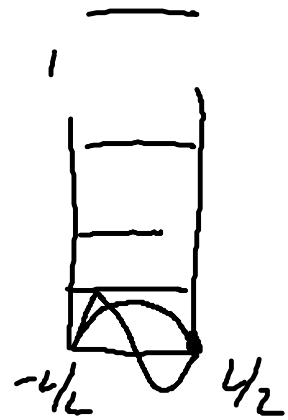
$$\Psi_\pm(x) = e^{\pm ikx}, \quad k = \sqrt{\frac{2m}{\hbar^2}E}$$

reell nur für $E > 0$

Normierung?: $\displaystyle\int_{-\infty}^{\infty} dx\, \underbrace{|\psi_k(x)|^2}_{1}$ existiert nicht.

Normierung auf 'endlichen Kasten' der Länge $L$

$$\Rightarrow \qquad \psi_k = \frac{1}{\sqrt{L}}\, e^{ikx}$$

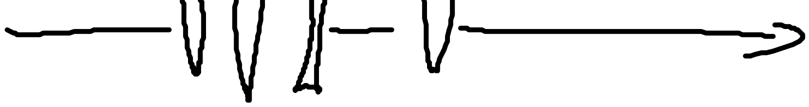


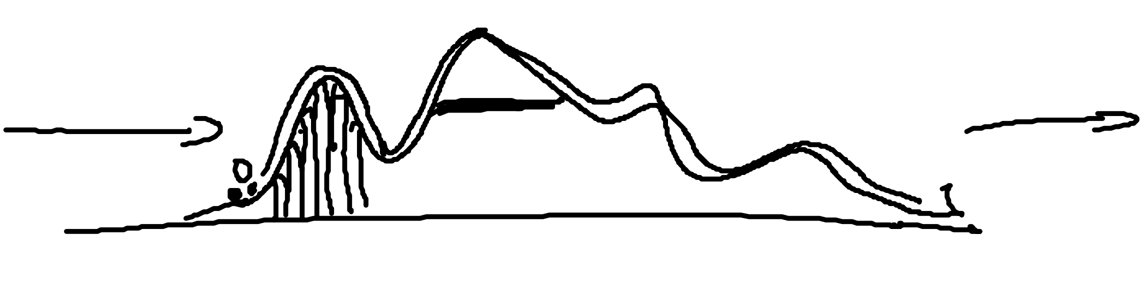

mit $\displaystyle\int_{-L/2}^{L/2} dx\, |\psi_k|^2 = 1$.

Randbedingungen: 1) feste RB

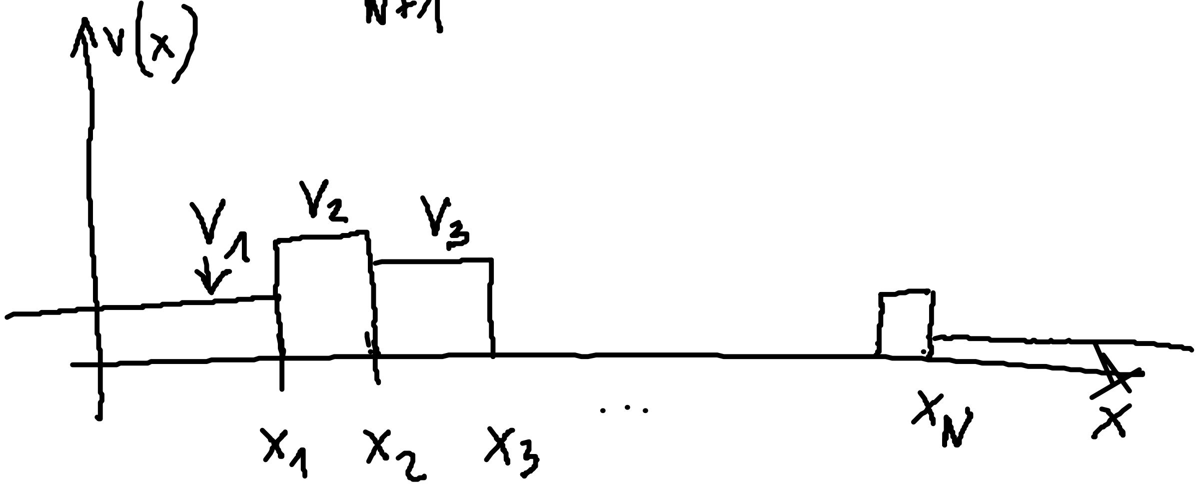
2) periodische RB $\Rightarrow$ $e^{ikL} = 1$

$$\Rightarrow k = \frac{2\pi}{L}\cdot n, \quad n = \pm 1, \pm 2, \dots$$



$$V(x) = \begin{cases} V_1 \\ V_2 \\ \vdots \\ V_{N+1} \end{cases} \qquad \psi(x) = \begin{cases} a_1 e^{ik_1 x} + b_1 e^{-ik_1 x}, & -\infty < x < x_1 \\ a_2 e^{ik_2 x} + b_2 e^{-ik_2 x}, & x_1 < x \leq x_2 \\ \vdots & \\ a_{N+1} e^{ik_{N+1} x} + b_{N+1} e^{-ik_{N+1} x}, & x_N < x < \infty \end{cases}$$




$$k_j \equiv \sqrt{\frac{2m}{\hbar^2}(E - V_j)}$$

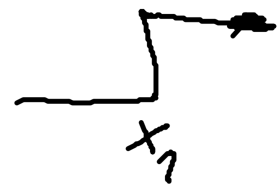
Wir lösen $\hat{H}\psi = E\psi$

mit $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \underbrace{V(x)}_{\text{stückweise konstant}}$.

Schritt 1: $\psi(x)$ und $\psi'(x)$ stetig bei $x = x_1$

$$-\Psi''(x) + V(x)\Psi(x) = E\Psi(x) \quad \Big/ \int_{x_1-\varepsilon}^{x_1+\varepsilon} dx$$

$$-\left\{\Psi'(x_1+\varepsilon) - \Psi'(x_1-\varepsilon)\right\} + \underbrace{\int_{x_1-\varepsilon}^{x_1+\varepsilon} dx\, V(x)\Psi(x)}_{|V(x)|<\infty \ \text{gilt}\ 0} = \underbrace{E\int_{x_1-\varepsilon}^{x_1+\varepsilon} dx\, \Psi(x)}_{0}$$




Ausnahme: $V(x) = V_0\,\delta(x_1)$, dann

$$\int_{x_1-\varepsilon}^{x_1+\varepsilon} dx\, V(x)\Psi(x) = V_0\Psi(x_1) \neq 0.$$

1) $$a_1 e^{ik_1x_1} + b_1 e^{-ik_1x_1} = a_2 e^{ik_2x_1} + b_2 e^{-ik_2x_1}$$

($\Psi(x)$ stetig)

$$ik_1\left\{a_1 e^{ik_1x_1} - b_1 e^{-ik_1x_1}\right\} = ik_2\left\{a_2 e^{ik_2x_1} - b_2 e^{-ik_2x_1}\right\}$$

| $\psi'(x)$ stetig

$$2)\ \{a_1 e^{ik_1x_1} - b_1 e^{-ik_1x_1}\} = \frac{k_2}{k_1}\{a_2 e^{ik_2x_1} - b_2 e^{-ik_2x_1}\}$$

$$a_1 = \frac{1}{2}\left(1+\frac{k_2}{k_1}\right) e^{i(k_2-k_1)x_1} a_2 + \frac{1}{2}\left(1-\frac{k_2}{k_1}\right) e^{-i(k_2+k_1)x_1} b_2$$

$$b_1 = \frac{1}{2}\left(1-\frac{k_2}{k_1}\right) e^{i(k_2+k_1)x_1} a_2 + \frac{1}{2}\left(1+\frac{k_2}{k_1}\right) e^{-i(k_2-k_1)x_1} b_2 .$$

Jetzt als Matrixgleichung $\quad \underline{u}_1 = T_1 \underline{u}_2$

$$\underline{u}_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$$

$$T_1 = \frac{1}{2k_1}\begin{pmatrix} (k_1+k_2)e^{i(k_2-k_1)x_1} & (k_1-k_2)e^{-i(k_1+k_2)x_1} \\ (k_1-k_2)e^{i(k_2+k_1)x_1} & (k_1+k_2)e^{-i(k_2-k_1)x_1} \end{pmatrix}$$


Schritt 2: $\quad \underline{u}_2 = T_2 \underline{u}_3 \quad$ etc

$$\underline{u}_1 = T_1 \underline{u}_2 = T_1 T_2 \underline{u}_3$$

$T_j$ heißt Transfer-Matrix.

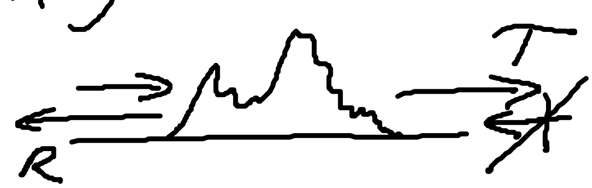
Damit $\quad \underline{u}_1 = \underline{\underline{M}}\, \underline{u}_{N+1}, \quad \underline{\underline{M}} = T_1 T_2 \cdots T_N.$

Transmissionskoeffizienten

$$T = \frac{j_{out}}{j_{in}} = \frac{\hbar k_{N+1} |a_{N+1}|^2}{\hbar k_1 \; |a_1|^2}$$

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} a_{N+1} \\ b_{N+1} \end{pmatrix}.$$

Streubedingung: $\quad b_{N+1} = 0$



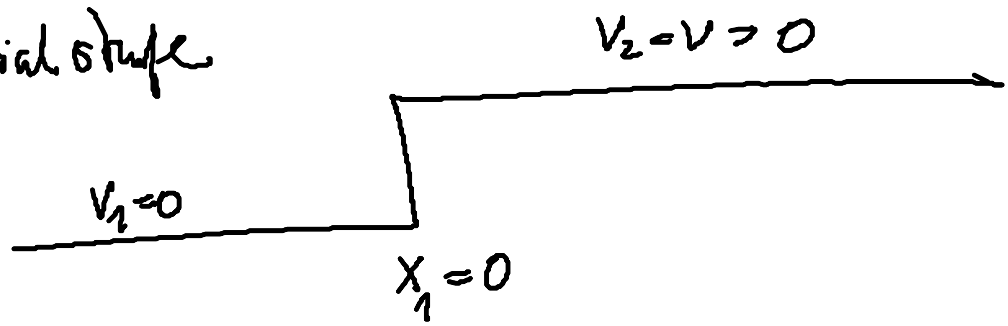
$$\Rightarrow \quad a_1 = M_{11}\, a_{N+1} + M_{12} \cdot 0$$

$$T = \frac{k_{N+1}}{k_1} \frac{1}{|M_{11}|^2}$$ Transmissionskoeffizient

Entsprechend $$R \equiv \left| \frac{b_1}{a_1} \right|^2 = \left| \frac{M_{21}}{M_{11}} \right|^2 .$$

Man zeigt: $$T + R = 1 .$$

Potentialstufe

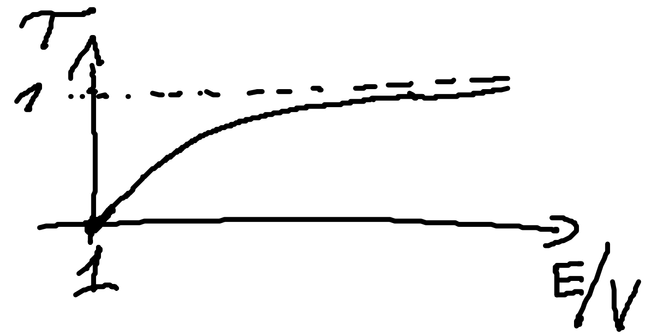


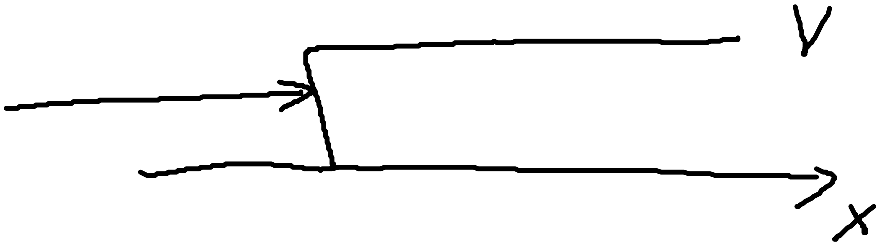

$$M_{11} = (T_1)_{11} = \frac{1}{2}\left(1 + \frac{k_2}{k_1}\right)$$

$$\Rightarrow \quad T = \frac{k_2}{k_1}\frac{1}{|M_{11}|^2} = \frac{k_2}{k_1}\frac{4k_1^2}{(k_1+k_2)^2}$$

$$k_1 = \sqrt{\frac{2m}{\hbar^2}E}\,, \quad k_2 = \sqrt{\frac{2m}{\hbar^2}(E-V)}$$

$$T = \frac{4\sqrt{1-V/E}}{\left(1+\sqrt{1-V/E}\right)^2}$$



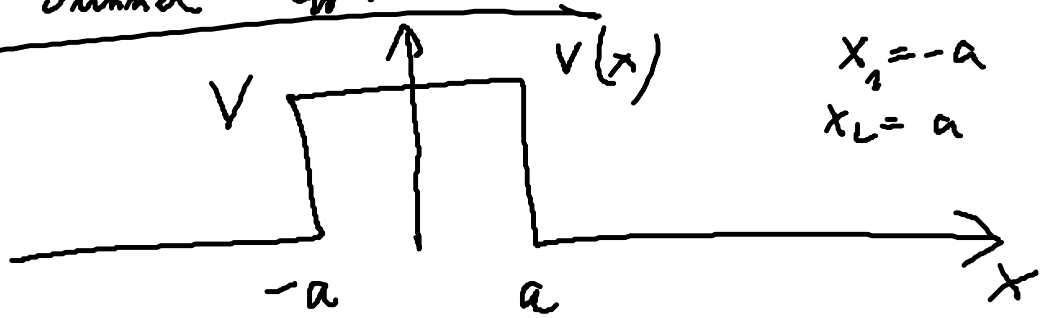




Für $E<V$ wird $k_2$ imaginär

$$\Rightarrow \quad R = \left|\frac{k_1 - i\kappa_2}{k_1 + i\kappa_2}\right|^2 = 1$$

$$i\kappa_2 = k_2\,, \quad \kappa_2 = \sqrt{\frac{2m}{\hbar^2}|E-V|}$$

## Der Tunnel-Effekt




$x_1 = -a$
$x_2 = a$

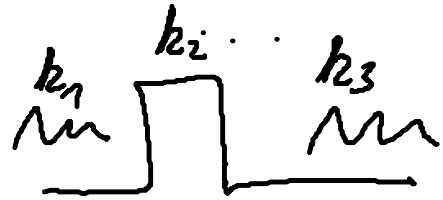
$$T_1 = \frac{1}{2}\begin{pmatrix} (1+r)\,e^{-i\delta_- a} & (1-r)\,e^{i\delta_+ a} \\ (1-r)\,e^{-i\delta_+ a} & (1+r)\,e^{i\delta_- a} \end{pmatrix}$$

$\delta_{\mp} \equiv k_2 \mp k_1$

$r \equiv k_2/k_1$

$$T_2 = \frac{1}{2}\begin{pmatrix} (1+1/r)\,e^{-i\delta_- a} & (1-1/r)\,e^{-i\delta_+ a} \\ (1-1/r)\,e^{i\delta_+ a} & (1-1/r)\,e^{i\delta_- a} \end{pmatrix}$$

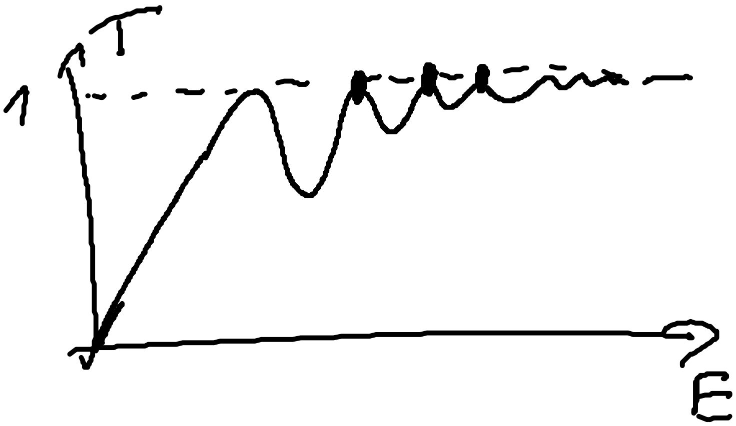
$$T_1T_2 = \frac{1}{4}\begin{pmatrix} \frac{(1+r)^2}{r}\,e^{-2i\delta_- a} - \frac{(1-r)^2}{r}\,e^{2i\delta_+ a} & \cdots \\ \cdots & \end{pmatrix}$$




$k_3 = k_1$

$$\Rightarrow \quad T = \frac{1}{1 + \frac{(k_1^2 - k_2^2)^2}{4k_1^2 k_2^2}\sin^2(2k_2 a)} \qquad E > V$$

$$T = \frac{1}{1 + \frac{(k_1^2 + \kappa_2^2)^2}{4k_1^2 \kappa_2^2}\sinh^2 2\kappa_2 a} \qquad E < V$$




$T = 1$ für $2\underbrace{k_2}_{2\pi/\lambda} a = n\pi$

an diesen Stellen

Resonanzbedingung

$$E_n = \frac{n^2\pi^2\hbar^2}{2m(2a)^2} + V$$

≙ Teilchen im Kasten.

Für $E < V$ : Tunneleffekt, endliche Transmissionswahrscheinlichkeit.