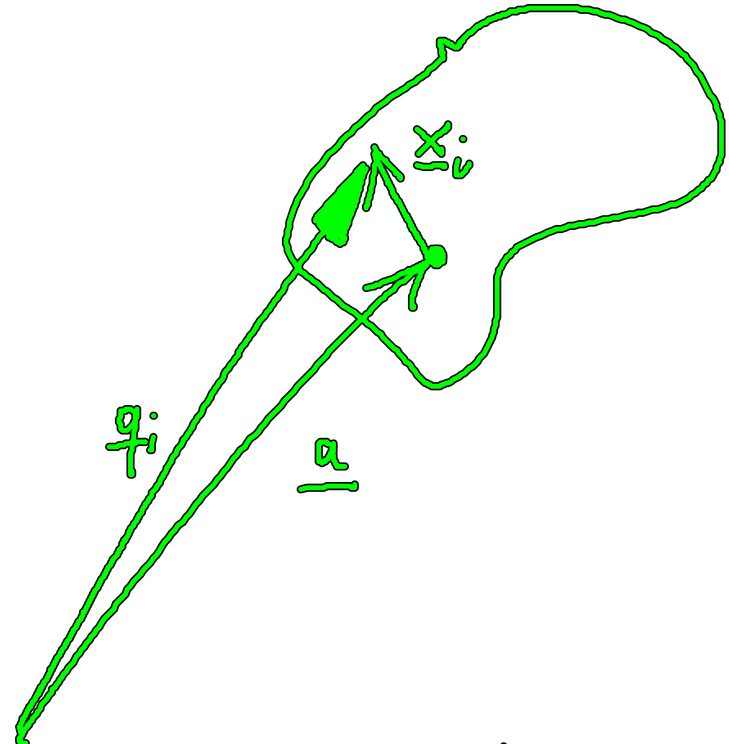
5.11.08



$$\ddot{\underline{a}} \overset{!}{=} 0$$

5.11.08

Drehimpuls

$$\underline{L} = \sum_{i=1}^{N} m_i \, q_i \times \dot{q}_i$$

$$\dot{\underline{L}} = \sum_{i=1}^{N} m_i \, q_i \times \ddot{q}_i =$$

$$= \sum_{i=1}^{N} m_i \, \underline{a} \times \ddot{q}_i + \sum_{i=1}^{N} m_i \, \underline{x}_i \times (\underbrace{\ddot{\underline{a}}}_{0} + \ddot{\underline{x}}_i)$$

$$= \sum m_i \, \underline{a} \times \ddot{q}_i + \dot{\underline{L}}_{rot}$$

$$\dot{\underline{L}}_{rot} \equiv \sum_{i=1}^{N} m_i \, \underline{x}_i \times \ddot{\underline{x}}_i$$

externe Kraft auf MP i

Drehmoment
$$\underline{M} = \sum_{i=1}^{N} \underline{q}_i \times \underline{F}_i^{ex} = \dot{\underline{L}}$$

$$\dot{\underline{L}} = \sum_i m_i \, \underline{a} \times \ddot{\underline{q}}_i + \dot{\underline{L}}_{rot} = \sum_{i=1}^{N} \underline{q}_i \times \underline{F}_i^{ex} = \sum_{i=1}^{N} \underline{a} \times \underline{F}_i^{ext} + \sum_{i=1}^{N} \underline{x}_i \times \underline{F}_i^{ext}$$

: Gesamtkraft $\underline{F}_{ges}^{ext} = \sum_{i=1}^{N} \underline{F}_i^{ext}$

$$\boxed{\dot{\underline{L}}_{rot}} = \sum_{i=1}^{N} \underline{x}_i \times \underline{F}_i^{ext} \equiv \boxed{\underline{M}_{rot}}$$

Jetzt ins K'-System transformieren

$$R\underline{M}'_{rot} = \underline{M}_{rot} = \dot{\underline{L}}_{rot} = R\left(\dot{\underline{L}}'_{rot} + \underline{\omega}' \times \underline{L}'_{rot}\right)$$

$\Uparrow$ durch die Rotation

$$\Rightarrow \boxed{\underline{M}'_{rot} = \dot{\underline{L}}'_{rot} + \underline{\omega}' \times \underline{L}'_{rot} \quad \text{im } K'\text{-System}}$$

Jetzt $\quad \underline{L}'_{rot} = \underline{\underline{\Theta}}' \, \underline{\omega}'$, $\underline{\underline{\Theta}}'$ ist zeitlich konstant in K'

$$\dot{\underline{L}}'_{rot} = \underline{\underline{\Theta}}' \, \dot{\underline{\omega}}'$$

einsetzen:

$$\boxed{\underline{M}'_{rot} = \underline{\underline{\Theta}}' \, \dot{\underline{\omega}}' + \underline{\omega}' \times \left(\underline{\underline{\Theta}}' \, \underline{\omega}' \right)}$$

$\underline{\omega}' = \underline{\omega}'(t)$ ist die gesuchte Größe

1. Fall: Der kräftefreie Kreisel $\underline{M}'_{rot} = 0$

Im HA System gilt

$$\underline{\underline{\Theta}}' \, \underline{\omega}' = \begin{pmatrix} \Theta_1' \omega_1' \\ \Theta_2' \omega_2' \\ \Theta_3' \omega_3' \end{pmatrix}$$

3. Komponente der Eulerschen Gleichung

$$\begin{array}{c} \quad + \qquad - \qquad + \\ -\Theta_3' \dot{\omega}_3' = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1' & \omega_2' & \omega_3' \\ \Theta_1' \omega_1' & \Theta_2' \omega_2' & \Theta_3' \omega_3' \end{vmatrix} = \omega_1' \omega_2' \left(\Theta_2' - \Theta_1' \right) \end{array}$$

+ zyklisch

$$\Theta_2' \dot{\omega}_2' = \omega_3' \omega_1' \left(\Theta_3' - \Theta_1' \right)$$

$$\Theta_1' \dot{\omega}_1' = \omega_2' \omega_3' \left(\Theta_2' - \Theta_3' \right)$$

Symmetrischer Kreisel : $\Theta_1' = \Theta_2' \,,\quad \Theta_3'$

Kugelkreisel : $\Theta_1' = \Theta_2' = \Theta_3'$

$$-\Theta_3' \dot{\omega}_3' = 0 \implies \omega_3' = \text{const} \; !$$

$$\Theta_1' \dot{\omega}_1' = \omega_2' \omega_3' \left(\Theta_1' - \Theta_3' \right)$$

$$\Theta_1' \dot{\omega}_2' = \omega_1' \omega_3' \left(\Theta_3' - \Theta_1' \right) \qquad \Big| \cdot i$$

$$\omega_\perp(t) = \omega_1(t) + i\, \omega_2(t)$$

$$\Theta_1' \left(\dot{\omega}_1' + i\, \dot{\omega}_2' \right) = \omega_3' \left(\Theta_3' - \Theta_1' \right) i \left(\omega_1' + i\, \omega_2' \right)$$

$$\implies \dot{\omega}_\perp(t) = \underbrace{\omega_3' \frac{\Theta_3' - \Theta_1'}{\Theta_1'}}_{= \omega_0'} \, i\, \omega_\perp(t)$$

$$\implies \text{vom Typ} \quad \dot{f}(t) = \alpha f(t)$$

Lösung $\quad \omega_\perp'(t) = \omega_\perp' e^{i \omega_0' t}$

jetzt zerlegen

$$\omega_1'(t) = \underline{\omega_\perp'} \cos(\omega_0' t)$$
$$\omega_2'(t) = \underline{\omega_\perp'} \sin(\omega_0' t)$$

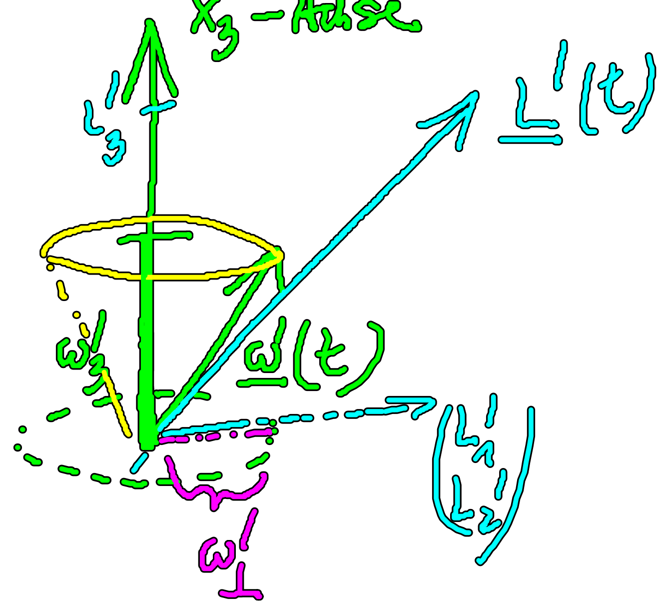
$$\left(\text{Euler:} \qquad e^{ix} = \cos x + i \sin x\right)$$

Der Betrag von $\underline{\omega}'(t)$ ist gegeben durch

$$|\underline{\omega}'(t)|^2 = \underline{\omega_\perp'}^2 + (\omega_3')^2 = \text{const}$$

geometr. Interpretation:

<u>Im K'-System:</u>



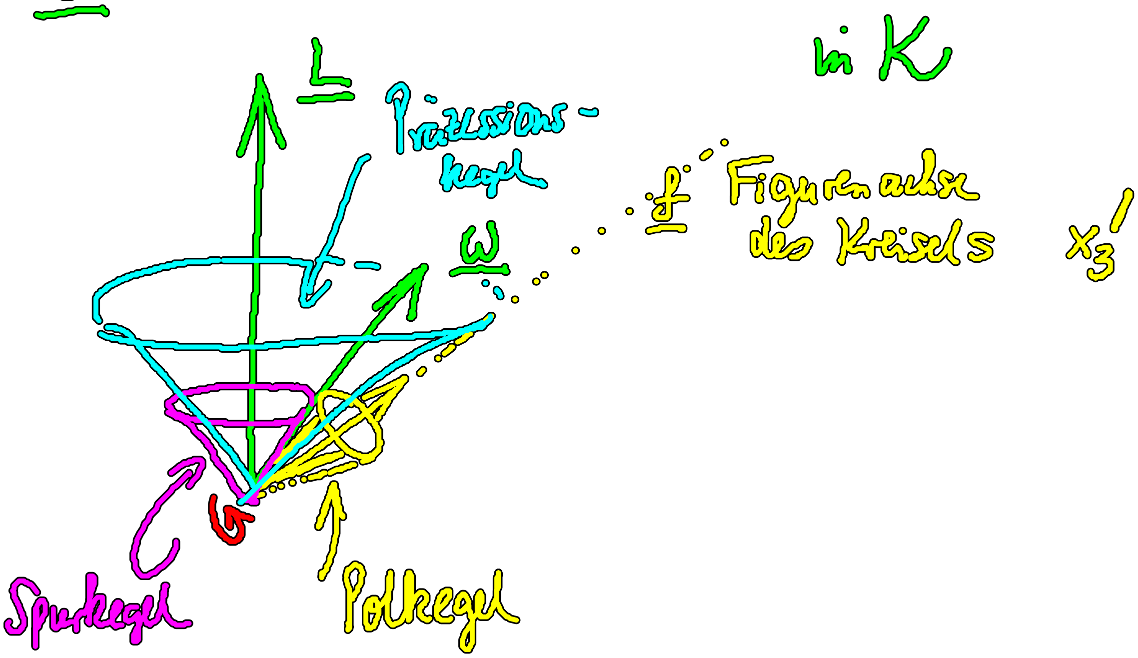
$$L_3' = \theta_3' \omega_3' = \text{const}$$
$$L_1'(t) = \theta_1' \omega_1'(t) = \theta_1' \underline{\omega_\perp'} \cos(\omega_0' t)$$
$$L_2'(t) = \theta_1' \omega_2'(t) = \theta_1' \underline{\omega_\perp'} \sin(\omega_0' t)$$

Es liegen also $\underline{e}_3'$, $\underline{\omega}'$, $\underline{L}'$ in einer gemeinsamen Ebene, die um die $\underline{e}_3'$-Achse rotiert mit der Winkelfrequenz ω_0'.

Wie sieht das in K aus?

Dort ist $\underline{L} = \text{const}$!

in K



"reguläre Präzession"

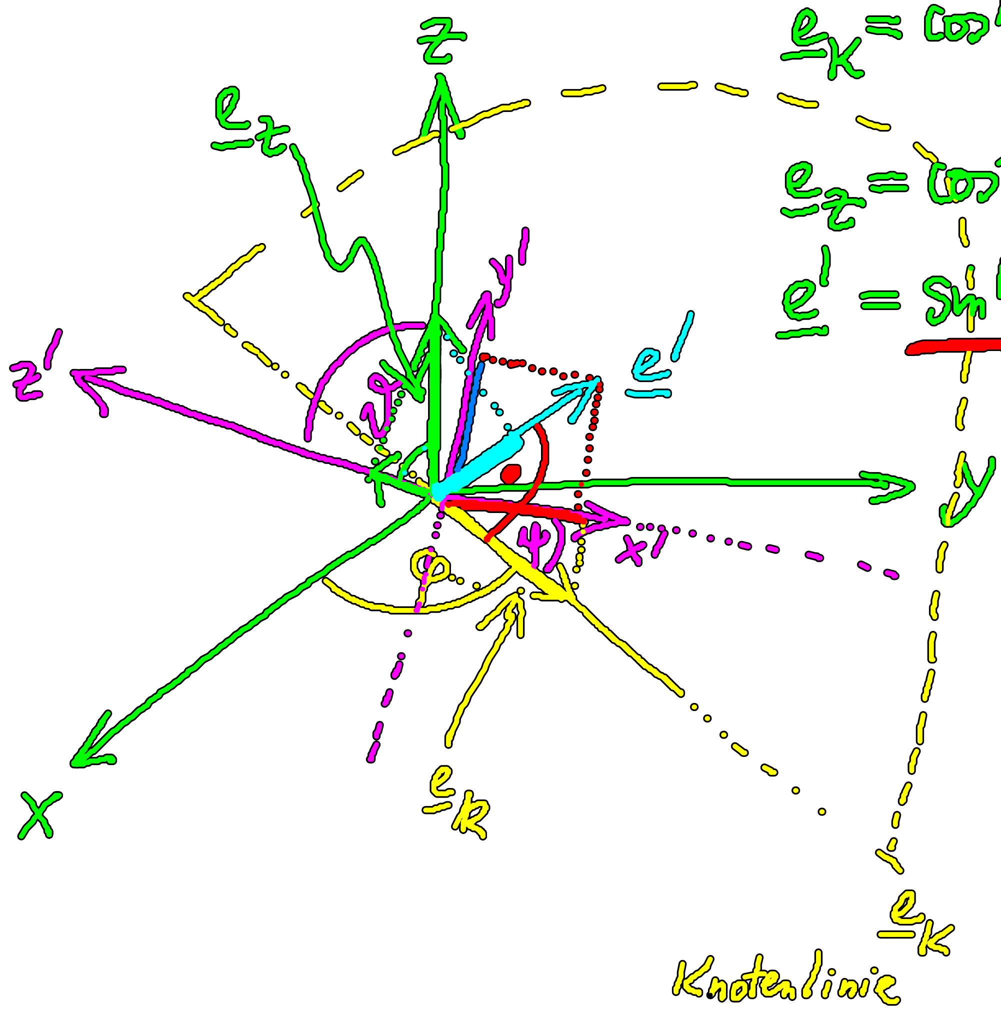
- Winkelgeschwindigkeit $\omega_0' = \omega_3' \dfrac{\Theta_3' - \Theta_1'}{\Theta_1'}$

Drei Eulerschen Winkel φ, ϑ, ψ

$$\underline{e}_K = \cos\psi\, \underline{e}'_x - \sin\psi\, \underline{e}'_y$$

$$\underline{e}_z = \cos\vartheta\, \underline{e}'_z + \sin\vartheta\, \underline{e}'$$

$$\underline{e}' = \sin\psi\, \underline{e}'_x + \cos\psi\, \underline{e}'_y$$



Damit

$$\underline{e}_z = \sin\psi \sin\vartheta\, \underline{e}'_x + \cos\psi \sin\vartheta\, \underline{e}'_y + \cos\vartheta\, \underline{e}'_z$$

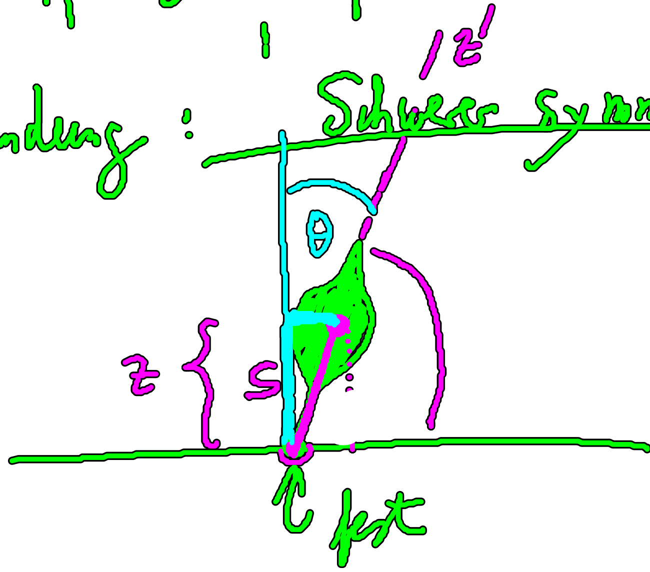
Für den Vektor der Winkelgeschwindigkeit gilt

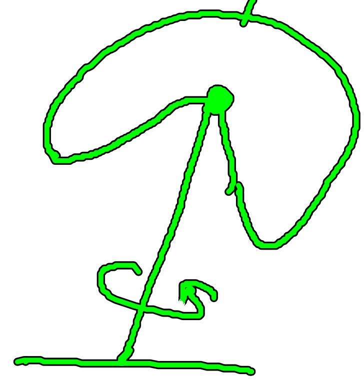
$$\underline{\omega} = \dot{\vartheta}\, \underline{e}_K + \dot{\varphi}\, \underline{e}_z + \dot{\psi}\, \underline{e}'_z$$

$$= \omega'_x\, \underline{e}'_x + \omega'_y\, \underline{e}'_y + \omega'_z\, \underline{e}'_z$$

$$\rightarrow \quad \left\| \begin{aligned} \omega_x' &= \dot{\varphi} \sin\vartheta \sin\psi + \dot{\vartheta} \cos\psi \\ \omega_y' &= \dot{\varphi} \sin\vartheta \cos\psi - \dot{\theta} \sin\psi \\ \omega_z' &= \dot{\varphi} \cos\vartheta + \dot{\psi} \end{aligned} \right\|$$

Anwendung: Schwerer symmetrischer Kreisel





$V = Mgz = Mg\, s \cos\vartheta$ potentielle Energie

$$T_{rot} = \frac{1}{2} \underline{\omega}' \underbrace{\underline{\underline{\Theta}}'}_{\text{sei diagonal}} \underline{\omega}' = \frac{1}{2} \left(\omega_x'^2 \Theta_1' + \omega_y'^2 \Theta_2' + \omega_z'^2 \Theta_3' \right)$$

symm. Kreisel: $\Theta_1' = \Theta_2'$

Jetzt Lagrange II mit den Koordinaten φ, ϑ, ψ

$$\Rightarrow L = T - V =$$

$$= \frac{1}{2}\left(\left(\dot\varphi^2 \sin^2\vartheta + \dot\vartheta^2\right)\Theta_1' + \left(\dot\varphi\cos\vartheta + \dot\psi\right)^2 \Theta_3' \right)$$

$$- M g s \cos\vartheta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = 0$$

L hängt nicht von φ und ψ ab
(nur von $\dot\varphi$, $\dot\psi$)

$$\Rightarrow \quad \frac{\partial L}{\partial \varphi} = 0 \qquad \varphi \text{ ist zyklisch}$$

$$\dot L_z \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} = 0 \Rightarrow L_z = \text{const}$$

$$\frac{\partial L}{\partial \psi} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot\psi} \equiv \dot L_z' = 0$$

$$\Rightarrow L_z' = \frac{\partial L}{\partial \dot\psi} = \text{const}$$

Damit haben wir bereits
zwei Konstanten der Bewegung.