6.1.09

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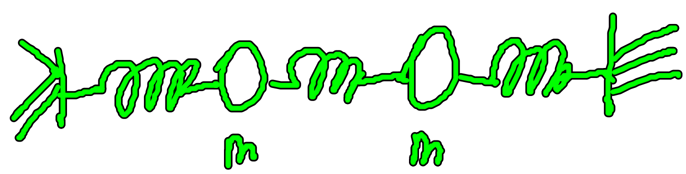
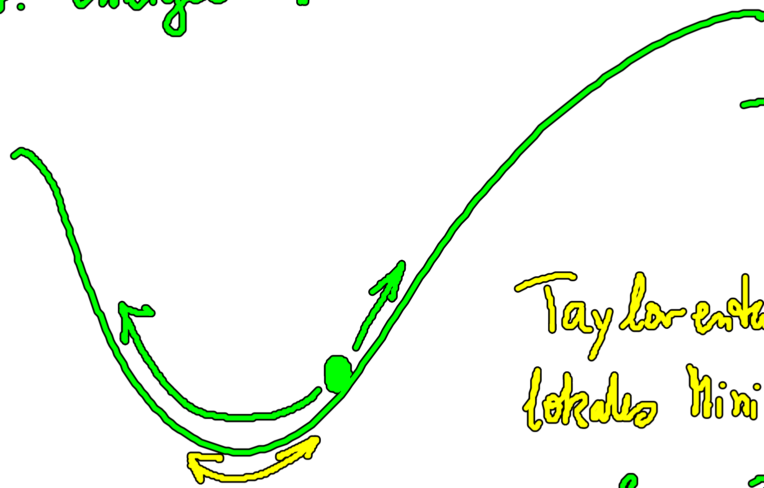
Kleine Schwingungen

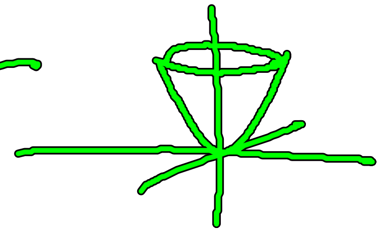
Lagrange , verallg. KO



$x_1, \dots, x_f$

pot. Energie V





Taylorentw. von $V(\underline{x})$ um lokales Minimum $\underline{x}_0$,

$$V(\underline{x}) = V(\underline{x}_0) + \frac{1}{2!}\sum_{i,j=1}^{f} \frac{\partial^2 V}{\partial x_i \partial x_j}\bigg|_{\underline{x}_0} (x_i - x_i^{(0)})(x_j - x_j^{(0)})$$

$$+ \ \ldots \qquad \underline{q} = \underline{x} - \underline{x}_0, \quad \underline{x} = (x_1, \ldots, x_f)^T$$

Man nähert

$$V(\underline{x}) \rightarrow \underbrace{V(\underline{x}_0)}_{\text{Konstante, wird weggelassen}} + \frac{1}{2}\underline{q}^T \underline{\underline{V}}^{(2)} \underline{q}$$

potentielle Energie $V(\underline{q}) = \frac{1}{2}\underline{q}^T \underline{\underline{V}}\,\underline{q}$

$f \times f$

kin. Energie hat Form

$$T(q,\dot{q}) = \frac{1}{2}\dot{q}^T \underline{\underline{T}}(q)\dot{q}$$

$f \times f$

z.B.

$$T = \frac{1}{2}(\dot{q}_1, \dot{q}_2)\begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$= \frac{1}{2}\dot{q}_1^2 m_1 + \frac{1}{2}\dot{q}_2^2 m_2 .$$

$\Rightarrow$ Lagrange-Funktion

$$L = T - V = \frac{1}{2}\dot{q}^T \underline{\underline{T}}\dot{q} - \frac{1}{2}q^T \underline{\underline{V}} q$$

Normalkoordinaten

linear Trafo $\quad q = A\underline{Q}, \quad A$ $f \times f$ Matrix (regulär)

$$A = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_f)$$

Spaltenvektoren $\underline{a}_i$: Vektoren der Normalmoden

z.B. $\quad \underline{Q} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow q = \underline{a}_1 .$

Jetzt Matrix A so wählen, dass die Lagrangefunktion entkoppelt:

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_f \end{pmatrix} \text{ Diagonalmatrix}$$

$$L = \frac{1}{2}\dot{q}^T T \dot{q} - \frac{1}{2} q^T V q$$

$$\overset{!}{=} \frac{1}{2}\dot{Q}^T \dot{Q} - \frac{1}{2} Q^T \underline{\underline{D}}\, Q$$

$$= \frac{1}{2}\sum_{i=1}^{f}\left(\dot{Q}_i^2 - \lambda_i Q_i^2\right) = \sum_{i=1}^{f} L_i$$

EL: $\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} - \frac{\partial L}{\partial Q_i} = 0; \quad L_i = \frac{1}{2}\left(\dot{Q}_i^2 - \lambda_i Q_i^2\right)$

$$\Rightarrow \boxed{\ddot{Q}_i(t) + \lambda_i Q_i(t) = 0}$$

DGL eines harmonischen Oszillators.

$$Q_i(t) = \alpha_i e^{i\omega_i t} + \beta_i e^{-i\omega_i t}$$

$$\omega_i^2 = \lambda_i$$

$$= a_i \cos \omega_i t + b_i \sin \omega_i t$$

$$L = \frac{1}{2}\dot{q}^T \underset{\sim}{T \dot{q}} - \frac{1}{2} q^T V q = \frac{1}{2}\underset{\sim}{\dot{Q}^T \dot{Q}} - \frac{1}{2} Q^T \underline{D} Q$$

$$q = AQ, \quad q^T = Q^T A^T$$

$$\Rightarrow L = \frac{1}{2}\dot{Q}^T \underbrace{A^T T A}\dot{Q} - \frac{1}{2} Q^T \underline{A^T V A} Q$$

$$\Rightarrow \left\| \begin{array}{l} A^T \underline{\underline{T}} A = \underline{\underline{E}} \;/\; D \quad f \times f \text{ Einheitsmatrix} \\ A^T \underline{\underline{V}} A = \underline{\underline{D}} \quad f \times f \text{ Diagonal matrix.} \end{array} \right.$$

Simultane Eigenwertprobleme:

$$D = A^T \underline{\underline{V}}\, \underline{\underline{A}} = A^T \underline{\underline{T}}\, \underline{\underline{A}}\, \underline{\underline{D}}$$

$$\underline{\underline{V}}\, \underline{\underline{A}} = \underline{\underline{T}}\, \underline{\underline{A}}\, \underline{\underline{D}}, \qquad \underline{\underline{A}} = (\underline{a}_1, \dots, \underline{a}_f)$$

$$\underline{\underline{V}} (\underline{a}_1, \dots, \underline{a}_f) = T (\underline{a}_1, \dots, \underline{a}_f) \underline{\underline{D}}$$

$$= T (\lambda_1 \underline{a}_1, \lambda_2 \underline{a}_2, \dots, \lambda_f \underline{a}_f)$$

$$(\underline{a}_1, \underline{a}_2) D = \left(\begin{pmatrix} \underline{a}_1 \end{pmatrix} \begin{pmatrix} \underline{a}_2 \end{pmatrix} \right) D = \left(\begin{pmatrix} \vdots \\ \underline{a}_1 \\ \vdots \end{pmatrix} \begin{pmatrix} \underline{a}_2 \end{pmatrix} \right) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Damit $\underline{\underline{V}}\,\underline{a}_i = \lambda_i \underline{\underline{T}}\,\underline{a}_i$ verallg. Eigenwertgleichung.

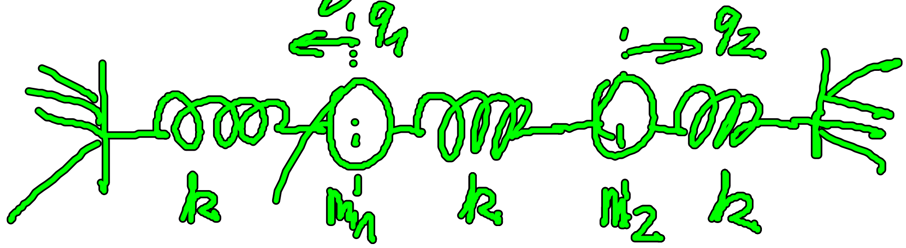
λ_i : Eigenwerte

Für nicht-triviale Lösungen der linearen Gleichungen

$(\underline{\underline{V}} - \lambda_i \underline{\underline{T}})\,\underline{a}_i = 0$ muß die Determinante

$$\det\left(\underline{\underline{V}} - \lambda_i \underline{\underline{T}}\right) = 0 .$$

Std.-
Beispiel:



$$L = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 - \left[\frac{1}{2} k q_1^2 + \frac{1}{2} k q_2^2 + \frac{1}{2} k (q_1 - q_2)^2\right]$$

$-2q_1q_2 = -q_1q_2 - q_1q_2$

also $\underline{\underline{T}} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ kin. Energie-Matrix

$$\underline{\underline{V}} = k \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2q_1 - q_2 \\ -q_1 + 2q_2 \end{pmatrix}$$

$$[\,] = \frac{1}{2}k\,(q_1, q_2)\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$(m_1 = m_2 = m)$

$$\det V - \lambda T = 0 \Rightarrow \begin{vmatrix} 2k - m\lambda & -k \\ -k & 2k - m\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2k - m\lambda)^2 - k^2 = 0$$

$$\Rightarrow \quad -2k + m\lambda = \pm k$$

$$\lambda_{1,2} = \frac{k}{m}, \; 3\frac{k}{m}$$

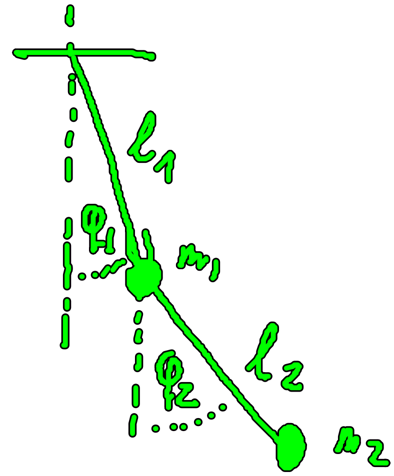
$$\omega_i^2 = \lambda_i \Rightarrow \quad \omega_1 = \sqrt{\frac{k}{m}} \; ; \; \omega_2 = \sqrt{3\frac{k}{m}}$$

<u>Eigenvektoren</u> $\quad (\underline{\underline{V}} - \lambda_1 \underline{\underline{T}})\,\underline{a}_1 = 0 \Rightarrow \underline{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ zu ω_1

$$(\underline{\underline{V}} - \lambda_2 \underline{\underline{T}})\,\underline{a}_2 = 0 \Rightarrow \underline{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

<u>Weiteres Beispiel</u> : Ebenes Doppelpendel.

Winkel $(q_1, q_2) = (\varphi_1, \varphi_2)$



$$T = \begin{pmatrix} (m_1+m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix}$$

$$V = \begin{pmatrix} (m_1+m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix}.$$

Normalform für Hamiltonfunktion

$$L = \frac{1}{2} \underline{\dot{Q}}^T \underline{\dot{Q}} - \frac{1}{2} \underline{Q}^T \underline{\underline{D}}\, \underline{Q}, \quad \underline{\underline{D}} = \begin{pmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_f^2 \end{pmatrix}$$

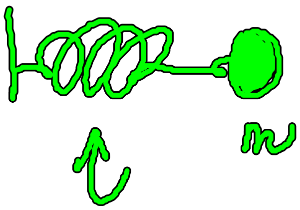
kan. Impulse $\quad P_i = \dfrac{\partial L}{\partial \dot{Q}_i} = \dot{Q}_i$

$$\rightarrow \quad H = \sum_{i=1}^{f} \left(\frac{1}{2} P_i^2 + \frac{1}{2} \omega_i^2 Q_i^2 \right)$$

$$= \sum_{i=1}^{f} H_i ; \qquad H_i : \text{ i-te harmonischer Oszillator}$$

alle entkoppelt.

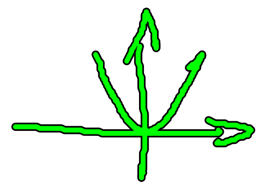
Ham. Oszillator



$$\cancel{m}\ddot{x} + \cancel{m}\omega^2 x = \underbrace{\cancel{m} f(t)}_{\text{äußere Kraft}}$$

$$\ddot{x} + \omega^2 x = f(t)$$

Masse m im Potential



$$V(x,t) = \frac{1}{2} m \omega^2 x^2 - x \cdot m f(t)$$

$$H(t) = \frac{p^2}{2m} + V(x,t)$$

$$\Rightarrow \quad \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x + m f(t)$$

$$\ddot{x} = \frac{\dot{p}}{m} = -\omega^2 x + f(t)$$

Reibungsterm:

$$\ddot{x} + \underbrace{\frac{1}{\tau}\dot{x}}_{\text{Dämpfungsterm}} + \omega_0^2 x = f(t), \quad \tau > 0$$

Als DGL-System (1. Ordnung).

$$\underline{y}'(t) = \underline{\underline{A}}\,\underline{y}(t) + \underline{b}(t)$$

$$\underline{y}(t) = \begin{pmatrix} x(t) \\ p(t)/m \end{pmatrix}; \quad \underline{b}(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -1/\tau \end{pmatrix}$$

Homogener Fall: Zeitentwicklung.

$$(*) \quad \underline{y}'(t) = A(t)\,\underline{y}(t); \quad \underline{y} = (y_1, \dots, y_n)$$

Wie beim Hamiltonschen Fluss $\phi^t : \underline{y}(0) \to \underline{y}(t)$

$$\underbrace{\underline{y}(t_0)}_{\text{Anfangsbed.}} \longrightarrow \underline{y}(t) = \underbrace{\underline{\underline{U}}(t, t_0)}_{\text{reelle } n\times n\text{-Matrix}}\,\underline{y}(t_0)$$

Einsetzen in $*$: $\frac{d}{dt}\left(\underline{\underline{U}}(t,t_0)\,\underline{y}(t_0)\right) - A(t)\,\underline{\underline{U}}(t,t_0)\,y(t_0)$

$$= \underbrace{\left(\frac{d}{dt}\underline{\underline{U}}(t,t_0) - A\,\underline{\underline{U}}(t,t_0)\right)}_{0}\underline{y}(t_0) = 0$$

0, es folgt

$\underline{y}(t_0) \to y(t_0)$

$$\frac{d}{dt}\underline{\underline{U}}(t,t_0) = \underline{A}(t)\,\underline{\underline{U}}(t,t_0)$$

$\underline{\underline{U}}(t_0, t_0) = E$ Einheitsmatrix

$U(t, t_0)$: Zeitentwicklungs-operator