

Energiedichte der el. magn. Welle:

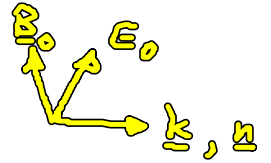
Sei \underline{E}_0 well $\Rightarrow \underline{E}(r,t) = E_0 \cos(kr - \omega t)$

$$\underline{B}(r,t) = B_0 \cos(kr - \omega t)$$

Energiedichte

$$w = \frac{\epsilon_0}{2} \underline{E}^2 + \frac{1}{2\mu_0} \underline{B}^2$$
$$= \frac{\epsilon_0}{2} \underline{E}^2 + \frac{1}{2\mu_0} \frac{1}{c^2} \underline{E}^2$$
$$= 2 \cdot \frac{\epsilon_0}{2} \underline{E}^2$$

$$\frac{1}{c^2} \underline{n} \times \underline{E}_0$$



$$\frac{1}{c^2} = \epsilon_0 \mu_0$$

Energiestromdichte: $\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$

$$= \frac{1}{\epsilon_0 \mu_0} \underline{E} \times (\underline{n} \times \underline{E})$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}} \underline{E}^2 \underline{n}$$

$$= c \epsilon_0 \underline{E}^2 \underline{n}$$

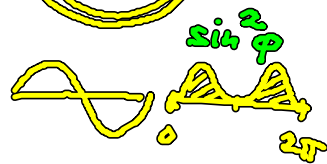
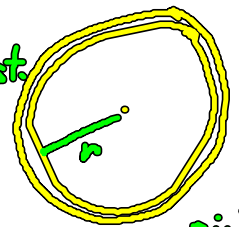
$$= c w \underline{n}, \quad \underline{n} = \frac{\underline{k}}{|\underline{k}|}$$

Kugelwelle: $\underline{E}(r,t) = \frac{1}{r} \underline{E}_0 e^{i(kr - \omega t)}$

Energie in Kugelschale mit Radius r u. Dicke dr :

$$W(r) = 4\pi r^2 dr \epsilon_0 \underline{E}^2 = 4\pi r^2 dr \epsilon_0 \frac{E_0^2}{r^2} \cdot \frac{1}{2} = \text{const.}$$

Raum/Zeitmittel
über $\exp i \dots$



4.2 Retardierte Potenziale

Aufgabe: Lösung der inhomog. Wellengl.

$$\boxed{\begin{aligned}\square \phi &= -\frac{1}{\epsilon_0} \rho \\ \square \underline{A} &= -\mu_0 \underline{j}\end{aligned}}$$

(Lorenz-Eichung)

zu vorgeg. erzeugenden Quellen $\rho(\underline{r}, t)$, $\underline{j}(\underline{r}, t)$
und Randbed. $\phi, \underline{A} \rightarrow 0$ für $r \rightarrow \infty$.

Methode: Green'sche Fkt. $G(\underline{r}-\underline{r}', t-t')$

vgl. Elektrostatik

$$\boxed{\square u(\underline{r}, t) = -f(\underline{r}, t)}$$

Fourier-Transform \Downarrow $u := \begin{cases} \phi \\ \underline{A} \end{cases}$
 $\hat{\square}^{-1} = -\hat{G} \Downarrow f := \begin{cases} \rho/\epsilon_0 \\ \mu_0 \underline{j} \end{cases}$

$$\boxed{\hat{u}(\underline{k}, \omega) = \hat{G} \cdot \hat{f}(\underline{k}, \omega)}$$

Rück-Transform \Downarrow

$$\boxed{u(\underline{r}, t) = \int_{\mathbb{R}^3} d^3r' \int_{-\infty}^{\infty} dt' G(\underline{r}-\underline{r}', t-t') f(\underline{r}', t')}$$

mit $\square G(\underline{r}-\underline{r}', t-t') = -\delta(\underline{r}-\underline{r}') \delta(t-t')$

$$\Delta \phi(\underline{r}) = -\frac{1}{\epsilon_0} \rho(\underline{r})$$

\Downarrow

$$\hat{\phi}(\underline{k}) = \hat{G} \hat{\rho}, \quad \hat{G} = \frac{1}{\epsilon_0 k^2}$$

\Downarrow

$$\phi(\underline{r}) = \int d^3r' G(\underline{r}-\underline{r}') \rho(\underline{r}')$$

$$\text{mit } G(\underline{r}-\underline{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\underline{r}-\underline{r}'|}$$

$$\Delta G(\underline{r}-\underline{r}') = -\frac{1}{\epsilon_0} \delta(\underline{r}-\underline{r}')$$

Kausalitätsbed.: $G(\underline{r}-\underline{r}', t-t') \stackrel{!}{=} 0$ für $t' > t$
damit $u(\underline{r}, t)$ nur von $f(\underline{r}', t')$ mit $t' < t$
beeinflusst wird.

Fourier-Transformation:

$$f(\underline{r}, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3 \underline{q} \int_{-\infty}^{\infty} d\omega \hat{f}(\underline{q}, \omega) e^{i(\underline{q} \cdot \underline{r} - \omega t)}$$

$$\hat{f}(\underline{q}, \omega) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3 \underline{r} \int_{-\infty}^{\infty} dt f(\underline{r}, t) e^{-i(\underline{q} \cdot \underline{r} - \omega t)}$$

Ebenso:

$$u(\underline{r}, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3 \underline{q} \int_{-\infty}^{\infty} d\omega \hat{u}(\underline{q}, \omega) e^{i(\underline{q} \cdot \underline{r} - \omega t)}$$

$$\begin{aligned} \square u(\underline{r}, t) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3 \underline{q} \int_{-\infty}^{\infty} d\omega \hat{u}(\underline{q}, \omega) \underbrace{\square e^{i(\underline{q} \cdot \underline{r} - \omega t)}}_{-(q^2 - \frac{\omega^2}{c^2}) e^{i(\underline{q} \cdot \underline{r} - \omega t)}} \\ &= -\frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} d^3 \underline{q} \int_{-\infty}^{\infty} d\omega \hat{f}(\underline{q}, \omega) e^{i(\underline{q} \cdot \underline{r} - \omega t)} \end{aligned}$$

$$\Rightarrow (q^2 - \frac{\omega^2}{c^2}) \hat{u}(\underline{q}, \omega) = \hat{f}(\underline{q}, \omega)$$

$$\Rightarrow \boxed{\hat{u}(\underline{q}, \omega) = \frac{\hat{f}(\underline{q}, \omega)}{q^2 - \frac{\omega^2}{c^2}}} \quad \text{d.h. } \hat{G} = \frac{1}{q^2 - \frac{\omega^2}{c^2}}$$

Rück-Trafo:

$$u(\underline{r}, t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} d^3 \underline{q} \int_{-\infty}^{\infty} d\omega \frac{e^{i(\underline{q} \cdot \underline{r} - \omega t)}}{q^2 - \frac{\omega^2}{c^2}} \int_{\mathbb{R}^3} d^3 \underline{r}' \int_{-\infty}^{\infty} dt' f(\underline{r}', t') e^{-i(\underline{q} \cdot \underline{r}' - \omega t')}$$

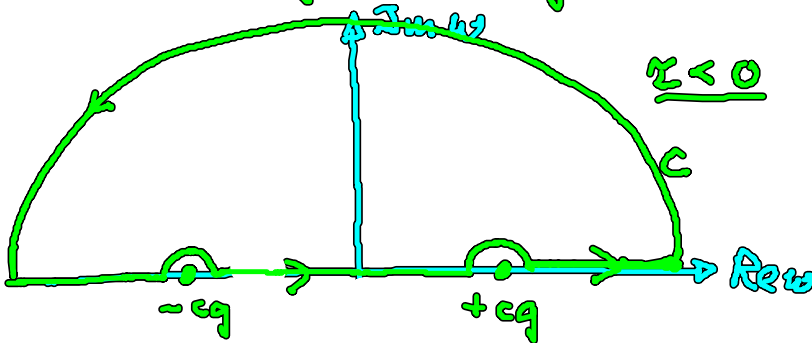
$$= \int_{\mathbb{R}^3} d^3 \underline{r}' \int_{-\infty}^{\infty} dt' \left\{ \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} d^3 \underline{q} \int_{-\infty}^{\infty} d\omega \frac{e^{i\underline{q} \cdot (\underline{r} - \underline{r}') - i\omega(t - t')}}{q^2 - \frac{\omega^2}{c^2}} \right\} f(\underline{r}', t')$$

$$G(\underline{r} - \underline{r}', t - t')$$

Berechnung der Green'schen Fkt. durch komplexe Integration

Integrand hat Pole bei $\omega = \pm cq$

Green'sche Fkt. wird eindeutig durch Festlegung des Integrationswegs um die Pole bestimmt



Integral über Halbkreis

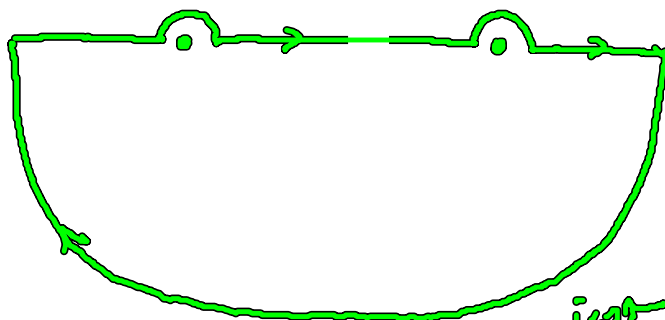
$$\omega = R e^{i\varphi} \quad 0 \leq \varphi \leq \pi$$

$$d\omega = R e^{i\varphi} i d\varphi$$

$$|e^{-i\omega t}| = e^{-R \sin(\varphi) t}$$

$\tau = t - t'$

$R \rightarrow \infty$
 $t > 0 < \tau < 0 \rightarrow 0$



$\tau > 0$: $\pi \leq \varphi \leq 2\pi$

$$|e^{-i\omega t}| = e^{-R \sin(\varphi) t} \xrightarrow{R \rightarrow \infty} 0$$

$$\Gamma(\underline{q}, \tau) := \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{q^2 - \frac{\omega^2}{c^2}} = \oint_C d\omega \frac{e^{-i\omega t}}{q^2 - \frac{\omega^2}{c^2}} = 2\pi i \sum_{\text{Pole}} \text{Res}$$

$\tau < 0$: keine Pole im Inneren von C

$$\Rightarrow \Gamma(\underline{q}, \tau) = 0 \Rightarrow G(\underbrace{\tau - \tau'}_s, \underbrace{t - t'}_t) = 0 \text{ für } t < t'$$

Dies ist die Kausalitätsbedingung!

$\tau > 0$: $\Gamma(\underline{q}, \tau) = -2\pi i \sum_{\omega = \pm cq} \text{Res} \frac{e^{-i\omega t}}{(-\frac{1}{c^2})(\omega - cq)(\omega + cq)}$

da $\oint_{\gamma} dz f(z) = 2\pi i \sum \text{Res} f(z)$, \leftrightarrow hier \odot
 math. pos. Sinn

$$\Gamma(\underline{q}, \tau) = 2\pi i c^2 \left(\frac{e^{-icq\tau}}{2cq} + \frac{e^{icq\tau}}{-2cq} \right)$$

$$G(\underline{s}, \tau) = \frac{c}{(2\pi)^3} \int d^3q e^{i\mathbf{q}\cdot\mathbf{s}} \left(\frac{e^{-icq\tau} - e^{icq\tau}}{-2iq} \right)$$

Auswertung in Kugelkoordin. : $d^3q = q^2 dq \sin\vartheta d\vartheta d\varphi$ $\frac{q}{r}$

$$q \cdot s = q s \cos\vartheta$$

$$G(\underline{x}, \tau) = \frac{c}{(2\pi)^3} \int_0^\infty dq q \frac{e^{-iq\tau} - e^{+iq\tau}}{-2i} \underbrace{\int_{-1}^1 d(\cos\vartheta) e^{iqs \cos\vartheta}}_{\frac{e^{iqs} - e^{-iqs}}{iqs}} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$\xi := cq$$

$$= \frac{1}{2(2\pi)^2 c} \int_0^\infty d\xi \left\{ \underbrace{e^{i(\tau - \frac{s}{c})\xi} + e^{-i(\tau - \frac{s}{c})\xi}}_{\delta(\tau - \frac{s}{c} - \xi)} \underbrace{- e^{i(\tau + \frac{s}{c})\xi} - e^{-i(\tau + \frac{s}{c})\xi}}_{\delta(\tau + \frac{s}{c} - \xi)} \right\}$$

Fourier:
 $\frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} = \delta(x)$

$$= \frac{1}{4\pi s} \left\{ \delta\left(\tau - \frac{s}{c}\right) - \delta\left(\tau + \frac{s}{c}\right) \right\}$$

0 für $\tau > 0$

Erq.:

$$G(\underline{r}-\underline{r}', t-t') = \begin{cases} \frac{1}{4\pi|\underline{r}-\underline{r}'|} \delta\left(t-t' - \frac{|\underline{r}-\underline{r}'|}{c}\right) & t > t' \\ 0 & t < t' \end{cases}$$

retardierte Green'sche Fkt.

(kausal)

