# Organisatorisches:

Mo., 15.02.2010 Nachklausur

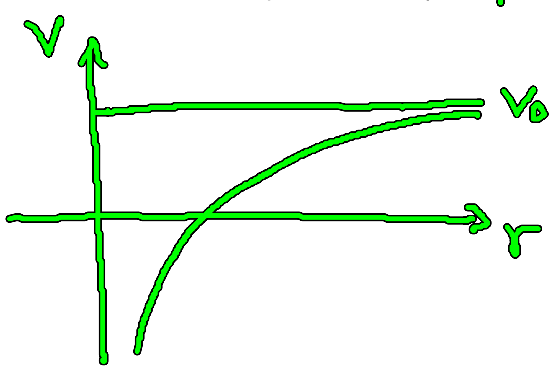
EW 201 ; 10 uhr

---

## Klausur:

**Aufgabe 1:** $V(r) = V_0 - \frac{\alpha}{r^3}$ ; $\alpha, V_0 > 0$

a) Skizze: $r := |\underline{r}| > 0$




b) Kraft:

$$\underline{F} = -\nabla_{\underline{r}} V$$

(i) $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ ; $\nabla_{\underline{r}} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$

(ii) $|\underline{r}| = (x^2+y^2+z^2)^{1/2}$

$$\Rightarrow \underline{F} = -\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \left[ -\frac{\alpha}{|\underline{r}|^3} \right]$$

$$= -\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \left( -\frac{\alpha}{(x^2+y^2+z^2)^{3/2}} \right)$$

$$= -\frac{3}{2}\alpha \cdot \frac{1}{(x^2+y^2+z^2)^{5/2}} \cdot \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$= -\frac{3\alpha}{(x^2+y^2+z^2)^{5/2}} \underline{r}$$

$$= -\frac{3a}{|\underline{r}|^4}\cdot\frac{\underline{r}}{|\underline{r}|} = -\frac{3a}{|\underline{r}|^4}\,\underline{e}_r$$ 1P.

c) Ebene Polarkoord.:

$$x = r\cos\phi \quad\Rightarrow\quad \dot r\cos\phi - r\dot\phi\sin\phi$$
$$y = r\sin\phi \quad\Rightarrow\quad \dot r\sin\phi + r\dot\phi\cos\phi$$ 1P.
$$z = z_0 \quad ; \quad z_0 = \text{const.}$$

$$\Rightarrow T = \frac{1}{2}m\left(\dot x^2+\dot y^2+\dot z^2\right) = \dots = \frac{m}{2}\left(\dot r^2+r^2\dot\phi^2\right)$$

$$\Rightarrow L = T - V = \underline{\underline{\frac{m}{2}\left(\dot r^2+r^2\dot\phi^2\right)+\frac{a}{r^3}-V_0}}$$ 2P.

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r \quad ; \quad p_\phi = \frac{\partial L}{\partial\dot\phi} = mr^2\dot\phi$$

$$\Rightarrow \dot r = \frac{p_r}{m} \qquad \dot\phi = \frac{p_\phi}{mr^2}$$ 1P.

$$H = \sum_{k=1}^{2}\dot q_k p_k - L$$
$$= \dot r p_r + \dot\phi p_\phi - \left\{\frac{m}{2}\dot r^2 + \frac{m}{2}\dot\phi^2 + \dots\right\}$$
$$\stackrel{!}{=} \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{a}{r^3} + V_0$$ 1P.

d)
$$\dot r = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$
$$\dot p_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{3a}{r^4}$$ 1P.
$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2}$$ 1P.
$$\dot p_\phi = -\frac{\partial H}{\partial\phi} = 0$$ 1P.

$\Rightarrow \phi$ ist zyklisch

Aufgabe 2: $V = -\frac{\alpha}{r}$

a) $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\Rightarrow T = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

$$V = \frac{\alpha}{(x^2+y^2+z^2)^{1/2}}$$

$$\Rightarrow H = T + V = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) - \frac{\alpha}{(x^2+y^2+z^2)^{1/2}}$$

1P

b) $\frac{d\vec{A}}{dt} = \{H, \vec{A}\} + \frac{\partial \vec{A}}{\partial t} \stackrel{!}{=} 0$

$$\vec{A} = \frac{1}{m}\vec{p} \times \vec{L} + V(r)\,\vec{r}$$

$$\frac{\partial \vec{A}}{\partial t} = 0 \quad \Rightarrow \quad \{H, \vec{A}\} = 0$$

$$\Leftrightarrow \frac{d\vec{A}}{dt} = 0$$

1P.

Allgem. Eigenschaft von Zentralpot.:
Bew. in der Ebene $\rightarrow$ $z = 0$ und $p_z = 0$

$$\vec{A} = \frac{1}{m}\begin{pmatrix} p_x \\ p_y \\ 0 \end{pmatrix} \times \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \right\} + V(r)\,\vec{r}$$

$$= \frac{1}{m}\begin{pmatrix} x p_y^2 - y p_x p_y \\ y p_x^2 - x p_x p_y \\ 0 \end{pmatrix} - \frac{\alpha}{(x^2+y^2)^{1/2}}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$A_z = 0$ !

2P.

$$H = \frac{1}{2m}(p_x^2 + p_y^2) - \frac{\alpha}{(x^2+y^2)^{1/2}}$$

$$A_x = \frac{1}{m}\left(x p_y^2 - y p_x p_y\right) - \frac{\alpha}{(x^2+y^2)^{1/2}}\, x$$

$$\{H, A_x\} = \sum_{k=1}^{3}\left(\frac{\partial H}{\partial q_k}\frac{\partial A_x}{\partial p_k} - \frac{\partial H}{\partial p_k}\frac{\partial A_x}{\partial q_k}\right)$$

$$= \frac{\partial H}{\partial x}\frac{\partial A_x}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial A_x}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial A_x}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial A_x}{\partial y}$$

$$\frac{\partial H}{\partial x} = \frac{\alpha x}{(x^2+y^2)^{3/2}} \; ; \; \frac{\partial H}{\partial y} = \frac{\alpha y}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} \; ; \; \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\frac{\partial A_x}{\partial x} = \frac{p_y^2}{m} - \frac{\alpha}{(x^2+y^2)^{1/2}} + \frac{\alpha x^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial A_x}{\partial y} = -\frac{p_x p_y}{m} + \frac{\alpha x y}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial A_x}{\partial p_x} = -\frac{y p_y}{m} \; ; \; \frac{\partial A_x}{\partial p_y} = \frac{2x p_y}{m} - \frac{y p_x}{m}$$

$$\Rightarrow \{H, A_x\} = \frac{\alpha p_x}{m(x^2+y^2)^{1/2}} - \frac{\alpha p_x}{m(x^2+y^2)^{3/2}}(x^2+y^2)$$

$$= \frac{\alpha p_x}{m(x^2+y^2)^{1/2}} - \frac{\alpha p_x}{m(x^2+y^2)^{1/2}}\,\frac{(x^2+y^2)}{(x^2+y^2)}$$

$$\stackrel{!}{=} 0 \qquad 3P.$$

→ Aus Symmetriegründen auch $\{H, A_y\} = 0$

$$\Rightarrow \{H, \vec{A}\} = 0 \Leftrightarrow \frac{d\vec{A}}{dt} = 0 \qquad 1P.$$

Aufgabe 3:

Schraubenlinie → Zylinder

a) (1) $x^2+y^2=R^2$ ; $R>0$

(2) $z=\frac{h}{2\pi}\phi$ ; $h>0$ 1P.

$b:=\frac{h}{2\pi}$

$\underline{r}(t)=\begin{pmatrix} R\cos\phi \\ R\sin\phi \\ b\phi \end{pmatrix}$ 1P.

b)

$$\underline{\underline{L=\frac{m}{2}(R^2+b^2)\dot{\phi}^2-mgb\phi}}$$ 1P.

c) $\frac{\partial L}{\partial \phi}-\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}=0$ 1P.

$\frac{\partial L}{\partial \phi}=-mgb$ ; $\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}=m(R^2+b^2)\ddot{\phi}$

$$\Rightarrow \underline{\underline{\ddot{\phi}=-\frac{gb}{R^2+b^2}}}$$ 1P.

d)

$$\underline{\underline{H=\frac{p_\phi^2}{2m(R^2+b^2)}+mgb\phi}}$$ 1P.

e) $\dot{\phi}=\frac{\partial H}{\partial p_\phi}$ ; $\dot{p}_\phi=-\frac{\partial H}{\partial \phi}$

$$\Rightarrow \ddot{\phi}=-\frac{gb}{R^2+b^2}$$ 2P.

f) 1. Integration nach $t$:

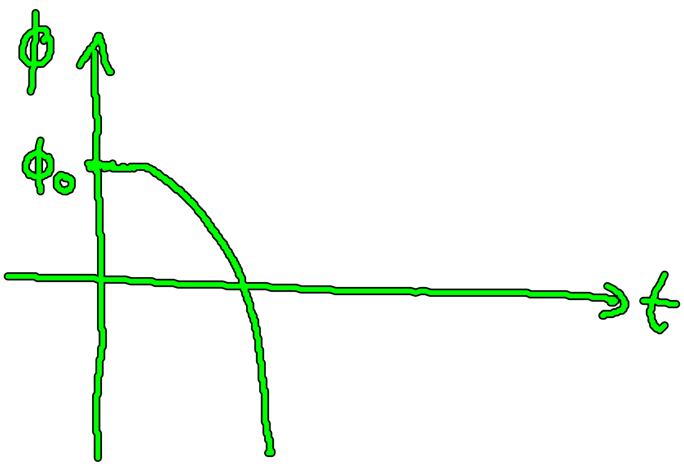
$$\dot{\phi} = -\frac{gb}{R^2+b^2}t + C_1$$

2. Integration:

$$\underline{\underline{\phi(t) = -\frac{1}{2}\frac{gb}{R^2+b^2}t^2 + C_1 t + C_2}}$$ 1P.

$$\Rightarrow \quad C_1 = \dot{\phi}(t=0) = \omega_0$$
$$C_2 = \phi(t=0) = \phi_0$$

g) Skizze:




1P.

Aufgabe 4:

a) $\rho(r) = \alpha r \quad ; \quad \alpha > 0$

$\underline{J}$ ist diagonal in allen Koord.-Systemen, die im Schwerpunkt der Kugel verankert sind.

$$J_{\mu\nu} = \int dr' \rho(r') \{ (\underline{r}')^2 \delta_{\mu\nu} - (\underline{r}')_\mu (\underline{r}')_\nu \}$$ 1P.

Kugelkoord.:

$$x = r\sin\vartheta\cos\phi$$
$$y = r\sin\vartheta\sin\phi \qquad dV = r^2\sin\vartheta\, dr\, d\vartheta\, d\phi$$
$$z = r\cos\vartheta$$

$$J_{11} = J_{22} = J_{33} \equiv J$$

$$J = J_{33} = \int_V dV\, \rho(r)\,(x^2+y^2)$$
$$= \alpha \int_0^R dr \int_0^\pi d\vartheta \int_0^{2\pi} d\phi\; r\{r^2\sin^2\vartheta\cos^2\phi + r^2\sin^2\vartheta\sin^2\phi\} \cdot r^2\sin\vartheta$$
$$= \alpha \int_0^R dr \int_0^\pi d\vartheta \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\; r^5\sin^3\vartheta$$
$$= 2\pi\alpha \int_0^R dr \int_0^\pi d\vartheta\; r^5 \sin\vartheta \underbrace{\sin^2\vartheta}_{(1-\cos^2\vartheta)}$$
$$= \frac{2\pi\alpha R^6}{6} \int_0^\pi d\vartheta\, \sin\vartheta\,(1-\cos^2\vartheta)$$
$$= \frac{\pi\alpha R^6}{3}\left[\int_0^\pi d\vartheta\,\sin\vartheta - \underbrace{\int_0^\pi d\vartheta\,\sin\cos^2\vartheta}_{= -\frac{1}{3}\cos^3\vartheta}\right]$$
$$= \frac{\pi\alpha R^6}{3}\left\{\underbrace{\left[-\cos\vartheta\right]_0^\pi}_{2} + \underbrace{\left[\frac{1}{3}\cos^3\vartheta\right]_0^\pi}_{-\frac{2}{3}}\right\}$$

$$= \frac{\pi \alpha R^6}{3} \cdot \frac{4}{3} = \frac{4}{9} \pi \alpha R^6$$ 2P.

$$M = \int_V dV \rho(r) = 4\pi\alpha \int_0^R r^3 dr$$

$$= 4\pi\alpha \left[ \frac{1}{4} r^4 \right]_0^R = \pi \alpha R^4$$

$$\Rightarrow \alpha(M) = \frac{M}{\pi R^4}$$ 1P.

$$\underline{\underline{J = \frac{4}{9} M R^2}}$$ 1P.

b) $$T_{rot} = \frac{1}{2} \underline{\omega}\, \underline{\underline{J}}\, \underline{\omega}$$ 1P.

$$= \frac{1}{2} \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} \omega_\mu J_{\mu\nu} \omega_\nu$$

$$= \frac{1}{2} J (\omega_1^2 + \omega_2^2 + \omega_3^2)$$

$$= \frac{1}{2} J \omega^2 \qquad \Big| \; J = \frac{4}{9} M R^2$$

$$\Rightarrow T_{rot} = \frac{2}{9} M (\omega R)^2$$

Abrollbedingung : $v = \omega R$

$$\Rightarrow T_{rot} = \frac{2}{9} M v^2$$ 1P.

c) Energieerhaltung :

$$E_{ges} = V_{pot}(h=h_0) = Mgh_0$$

Am untersten Punkt Geschw. maximal $\rightarrow v_{max}$
und dann :

$$E_{ges} = T_{trans.} + T_{rot} + V_{pot}(h=0)$$

$$Mgh_0 = \frac{1}{2} M v_{max}^2 + \frac{2}{9} M v_{max}^2 + 0$$

$$gh_0 = \frac{13}{18} v_{max}^2$$

$$\Rightarrow \underline{\underline{v_{max} = \sqrt{\frac{18}{13} g h_0}}}$$ 1P.

$$h_0 = l \sin\beta \Rightarrow v_{max} = \sqrt{\frac{18}{13} g l \sin\beta}$$