English summary: 4.3 Canonical transformation

idea: change of coordinates $(\underline{q},\underline{p},t)\longrightarrow(\underline{Q},\underline{P},t)$ such that the Hamiltonian remains form invariant, i.e., the action integral over the Lagrangian must be stationary:

$$\delta\int_{t_1}^{t_2} dt\, L = \delta\int_{t_1}^{t_2} dt\left\{\sum_{i=1}^{f} p_i\dot{q}_i - H(\underline{q},\underline{p},t)\right\}=0 \qquad (1)$$

and

$$\delta\int_{t_1}^{t_2} dt\left\{\sum_{i=1}^{f} P_i\dot{Q}_i - \bar{H}(\underline{Q},\underline{P},t)\right\}=0 \qquad (2)$$

Claim: (1) and (2) are equivalent if:

$$\sum_{i=1}^{f} p_i\dot{q}_i - H(\underline{q},\underline{p},t) = \sum_{i=1}^{f} P_i\dot{Q}_i - \bar{H}(\underline{Q},\underline{P},t) + \frac{d}{dt}M_1(\underline{q},\underline{Q},t) \qquad (3)$$

$M_1(\underline{q},\underline{Q},t)$: generating function of the canonical transformation

Fortsetzung von Kapitel 4.3

Beweis, dass (1) und (2) äquivalent sind:

a) $$\frac{d}{dt}M_1(\underline{q},\underline{Q},t) = \sum_{i=1}^{f}\left(\frac{\partial M_1}{\partial q_i}\dot{q}_i + \frac{\partial M_1}{\partial Q_i}\dot{Q}_i\right) + \frac{\partial M_1}{\partial t}$$

$$\overset{(3)}{\Longrightarrow} \sum_{i=1}^{f}\left(p_i - \frac{\partial M_1}{\partial q_i}\right)\dot{q}_i = \sum_{i=1}^{f}\left(P_i + \frac{\partial M_1}{\partial Q_i}\right)\dot{Q}_i + H - \bar{H} + \frac{\partial M_1}{\partial t}$$

Da $\underline{q}$ und $\underline{Q}$ unabhängige Variablen sind, folgt somit aus (3):

$$\boxed{p_i = \frac{\partial M_1}{\partial q_i}\quad,\quad P_i = -\frac{\partial M_1}{\partial Q_i}\quad,\quad \bar{H} = H + \frac{\partial M_1}{\partial t}}$$

Damit ist die kanonische Transformation durch $M_1(\underline{q},\underline{Q},t)$ eindeutig bestimmt: $p_i = \frac{\partial M_1}{\partial q_i} \overset{\text{Umkehrung}}{\Longrightarrow} Q_i(\underline{q},\underline{p},t)$ falls $\det\left(\frac{\partial^2 M_1}{\partial q_i\,\partial Q_i}\right)\neq 0$

$$P_i = -\frac{\partial M_1(\underline{q},\underline{Q},t)}{\partial Q_i} = -\frac{\partial M_1(\underline{q},\underline{Q}(\underline{q},\underline{p},t),t)}{\partial Q_i} = P_i(\underline{q},\underline{p},t)$$

Umkehrtransformation: $q_i = q_i(\underline{Q},\underline{P},t)$

eingesetzt in $p_i = \frac{\partial M_1}{\partial q_i} \Longrightarrow p_i = p_i(\underline{Q},\underline{P},t)$

b) Wir zeigen nun, dass (1) und (2) äquivalent sind:

$$(1) \Leftrightarrow 0 \overset{!}{=} \delta\int_{t_1}^{t_2} dt\left\{\sum_i p_i\dot q_i - H\right\} \underset{(3)}{=} \delta\int_{t_1}^{t_2} dt\left\{\sum_i P_i\dot Q_i - \bar H + \frac{d}{dt}M_1\right\}$$

$$= \underbrace{\delta\int_{t_1}^{t_2} dt\left\{\sum_i P_i\dot Q_i - \bar H\right\}}_{=\int_{t_1}^{t_2} dt \sum_i\left(\delta P_i\dot Q_i + P_i\delta\dot Q_i - \frac{\partial \bar H}{\partial Q_i}\delta Q_i - \frac{\partial\bar H}{\partial P_i}\delta P_i\right)} + \underbrace{\delta\left\{M_1(\underline{q}(t_2),\underline{Q}(t_2),t_2) - M_1(\underline{q}(t_1),\underline{Q}(t_1),t_1)\right\}}_{=\sum_i\left(\underbrace{\frac{\partial M_1}{\partial q_i}\delta q_i\Big|_{t_1}^{t_2}}_{=0,\ \text{da } \delta q(t_1)=0,\ \delta q(t_2)=0} + \underbrace{\frac{\partial M_1}{\partial Q_i}\delta Q_i\Big|_{t_1}^{t_2}}_{\text{i.a.} \neq 0}\right)}$$

Mit $\int_{t_1}^{t_2} dt\, P_i\delta\dot Q_i = P_i\delta Q_i\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \dot P_i \delta Q_i$ folgt:

$$0 = \delta\int_{t_1}^{t_2} dt\, L = \sum_i \underbrace{\left(P_i + \frac{\partial M_1}{\partial Q_i}\right)}_{=0 \text{ Konstruktion von } M_1}\delta Q_i\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\sum_i\left\{\underbrace{\left(\dot Q_i - \frac{\partial\bar H}{\partial P_i}\right)}_{\overset{!}{=}0}\delta P_i - \underbrace{\left(\dot P_i + \frac{\partial \bar H}{\partial Q_i}\right)}_{\overset{!}{=}0}\delta Q_i\right\}$$

$\delta P_i, \delta Q_i$ unabhängig:

gilt für: $\dot Q_i = \frac{\partial \bar H}{\partial P_i}$ und $\dot P_i = -\frac{\partial \bar H}{\partial Q_i}$

Hamilton-Gleichungen für $\bar H(\underline{Q},\underline{P},t)$ □

Beispiele für $M_1(q,Q)$ siehe ÜB 11 Aufgaben 26+27

A26: $M_1(q,Q) = k\frac{q^2}{2\tan Q}$ A27: $M_1(\underline{q},\underline{Q}) = \sum_{i=1}^{f} q_i Q_i$

Äquivalente Formulierung der erzeugenden Funktion:

Legendre-Transformation von $M_1(\underline{q},\underline{Q},t)$: (für $\underline{Q}$)

$$M_2(\underline{q},\underline{P},t) = M_1(\underline{q},\underline{Q},t) - \sum_{i=1}^{f} \frac{\partial M_1}{\partial Q_i} Q_i$$

Aus (3) folgt $\left(\sum_i (p_i\dot q_i - P_i\dot Q_i) - (H - \bar H) = \frac{d}{dt} M_1\right)$ mit

$$\frac{d}{dt}H_1 = \frac{d}{dt}\left(H_2(q,P,t) - \sum_i P_i Q_i\right) = \sum_i\left(\frac{\partial H_2}{\partial q_i}\dot q_i + \frac{\partial H_2}{\partial P_i}\dot P_i - Q_i\dot P_i - P_i\dot Q_i\right) + \frac{\partial H_2}{\partial t}$$

$$P_i = -\frac{\partial H_1}{\partial Q_i}$$

$$\sum_i\left\{\left(p_i - \frac{\partial H_2}{\partial q_i}\right)\dot q_i + \left(Q_i - \frac{\partial H_2}{\partial P_i}\right)\dot P_i + \cancel{(P_i - P_i)\dot Q_i}\right\} = \left(H - \bar H + \frac{\partial H_2}{\partial t}\right)$$

also (q, P unabhängig):

$$\boxed{p_i = \frac{\partial H_2}{\partial q_i}\ ,\quad Q_i = \frac{\partial H_2}{\partial P_i}\ ,\quad \bar H = H + \frac{\partial H_2}{\partial t}}$$

Beispiel: $H_2(q,P,t) = \sum_{i=1}^{f} q_i P_i$

$$\Rightarrow p_i = P_i = \frac{\partial H_2}{\partial q_i},\quad Q_i = \frac{\partial H_2}{\partial P_i} = q_i \qquad \text{(identische Transformation)}$$

(in komplizierteren Fällen $q_i = q_i(Q,P,t)$, $P_i = P_i(Q,P,t)$)

Analog: $H_3(p,Q,t) = H_1(q,Q,t) - \sum_i \frac{\partial H_1}{\partial q_i} q_i$

$$\Rightarrow q_i = -\frac{\partial H_3}{\partial p_i}\ ,\quad P_i = -\frac{\partial H_3}{\partial Q_i}$$

$$H_4(p,P,t) = H_1(q,Q,t) - \sum_i\left(\frac{\partial H_1}{\partial q_i}q_i + \frac{\partial H_1}{\partial Q_i}Q_i\right)$$

$$\Rightarrow q_i = -\frac{\partial H_4}{\partial p_i}\ ,\quad Q_i = \frac{\partial H_4}{\partial P_i}$$

4.4 Symplektische Struktur des Phasenraums

Da die kanonischen Transformationen generalisierte Koordinaten und Impulse ineinander transformieren können, sollten q und p nicht voneinander ausgezeichnet sein. Um diese Symmetrie des kanonischen Formalismus sichtbar zu machen, führen wir in diesem Abschnitt eine neue Notation ein.

Zunächst $f=1$: $\underline{x} := \begin{pmatrix} q \\ p \end{pmatrix}$ Vektor im Phasenraum

$$\underline{H}_x := \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} \quad \text{Ableitungsvektor}$$

$$\underline{\underline{J}} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ Metrik im Phasenraum (metrischer Tensor)

"schiefsymmetrische Matrix": $\underline{\underline{J}}^T = -\underline{\underline{J}}$

kanonische Gleichungen: $\boxed{\underline{\dot{x}} = \underline{\underline{J}}\, \underline{H}_x} \Leftrightarrow \begin{array}{l} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{array}$

$$\Leftrightarrow \boxed{-\underline{\underline{J}}\, \underline{\dot{x}} = \underline{H}_x}$$

Es gilt: $\underline{\underline{J}}^2 = -\mathbb{1}$, $\underline{\underline{J}}^{-1} = \underline{\underline{J}}^T = -\underline{\underline{J}}$

Beliebige Anzahl von Freiheitsgraden:

$$\underline{x} = \begin{pmatrix} q_1 \\ \vdots \\ q_f \\ p_1 \\ \vdots \\ p_f \end{pmatrix}, \quad \underline{H}_x = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_f} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_f} \end{pmatrix}, \quad \underline{\underline{J}} = \begin{array}{c} \begin{pmatrix} \underline{0} & \underline{\mathbb{1}} \\ -\underline{\mathbb{1}} & \underline{0} \end{pmatrix} \end{array} \begin{array}{l} \}f \\ \}f \end{array}$$

mit $\mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $f \times f$ Matrix

kanonischen Gleichungen: $\underline{\dot{x}} = \underline{\underline{J}}\, \underline{H}_x \Leftrightarrow -\underline{\underline{J}}\, \underline{\dot{x}} = \underline{H}_x$ $\quad \begin{array}{l} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{array}$