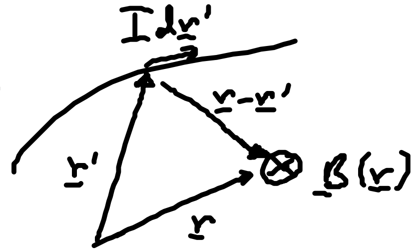
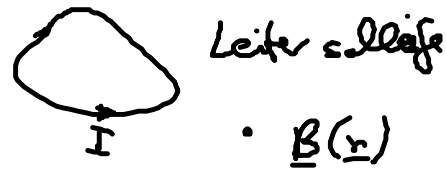
## 5.2 Leiter & Magnetfeld

$$d\underline{B} = \frac{\mu_0}{4\pi} I \, d\underline{r}' \times \frac{\underline{r} - \underline{r}'}{|\underline{r} - \underline{r}'|^3} \quad (5.10)$$




$$\underline{\underline{B}}(\underline{r}) = \oint d\underline{B}(\underline{r})$$

Integration über Leiterschleife



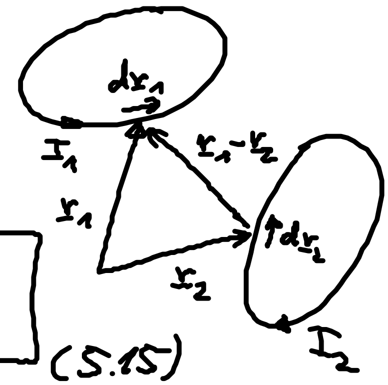

- Kraft zwischen Leiterschleifen:

  [Kraft auf Leiter:

  $$\underline{F} = I \int_C d\underline{r} \times \underline{B} \quad (5.9)]$$

von $2 \to 1$:

$$\boxed{\underline{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\underline{r}_1 - \underline{r}_2}{|\underline{r}_1 - \underline{r}_2|^3} \, d\underline{r}_1 \cdot d\underline{r}_2} \quad (5.15)$$

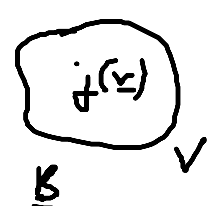



Beweis: Übung

## 5.3 Grundgleichungen der Magnetostatik

- Übergang von Stromelement $I d\underline{r}$ $\longrightarrow$ Stromdichte $\underline{j}(\underline{r})$

  mit $I d\underline{r} = \underline{j}(\underline{r}) d^3\underline{r}$




- Kraft auf Stromgebiet:

$$\xrightarrow{(5.9)} \boxed{\underline{F}(\underline{r}) = \int \underline{f}(\underline{r}) \, d^3 r = \int \underline{j}(\underline{r}) \times \underline{B}(\underline{r}) \, d^3 r} \quad (5.16)$$

NB: Punktladung $q$

$$\mathbf{j} = q\mathbf{v}\,\delta(\mathbf{r}-\mathbf{r}(t)) \longrightarrow \boxed{\mathbf{F}(\mathbf{r}) = q\left[\underbrace{\mathbf{v}\times\mathbf{B}(\mathbf{r},t)}_{\text{... Lorentzkraft!}} + \mathbf{E}(\mathbf{r},t)\right]} \quad (5.17)$$

zusätzlich elektr. Feld!

- magnet. Induktion:

$$\xrightarrow{(5.10)} \boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{j}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3r'} \quad (5.18)$$

NB: vgl. $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{r}')\, \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3r'$

- Vektorpotential: mit $\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}$

$$(5.18) \longrightarrow \boxed{\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) \quad \text{mit} \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'} \quad (5.19)$$

$$\longrightarrow \operatorname{div}\mathbf{B} = 0$$

- Wirbel der magnet. Induktion:

(i) $\operatorname{rot}\mathbf{B} = \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

$$= \underbrace{\frac{\mu_0}{4\pi}\nabla\int \mathbf{j}(\mathbf{r}')\cdot\underbrace{\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}}_{-\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}}\, d^3r'}_{=0,\ \text{da}\ \nabla'\cdot\mathbf{j}(\mathbf{r}')=0} - \frac{\mu_0}{4\pi}\int \mathbf{j}(\mathbf{r}')\underbrace{\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}}_{-4\pi\,\delta(\mathbf{r}-\mathbf{r}')}\, d^3r'$$

part. Integ. & Oberflächenterm $= 0$

$$= \mu_0\,\mathbf{j}(\mathbf{r}) \quad (5.21a)$$

$\longrightarrow$ Feldgleichungen der Magnetostatik

$$\boxed{\begin{aligned} &\operatorname{div}\mathbf{B} = 0 && (5.20) \\ &\operatorname{rot}\mathbf{B} = \mu_0\,\mathbf{j}(\mathbf{r}) && (5.21) \end{aligned}}$$

(5.20) ... keine magnet. Monopole!

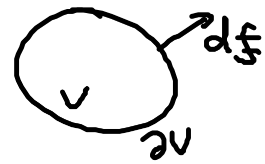
(5.21) ... Ampèresches Gesetz, Ströme = Wirbel von $\mathbf{B}$

NB: mit $\underline{B} = \mu_0 \underline{H}$ (im Vakuum)

$$(5.21) \longrightarrow \boxed{\text{rot}\, \underline{H} = \underline{j}(\underline{r})} \quad (5.22)$$

· integrale Formulierungen:

(1) $\int_V \text{div}\, \underline{B}\, d^3r = 0$

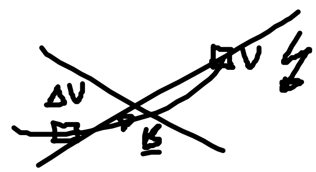



$$\xrightarrow{\text{Gauß}} \boxed{\oint_{\partial V} \underline{B} \cdot d\underline{f} = 0} \quad (5.23)$$

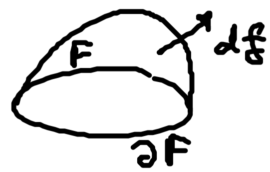
... magnetischer Fluß aus V = 0

NB: „Anzahl der Feldlinien in = aus V heraus"

⟶ Feldlinien sind geschlossen!




(2) $\int_F \text{rot}\, \underline{B} \cdot d\underline{f} \overset{\text{Stokes}}{=} \oint_{\partial F} \underline{B} \cdot d\underline{r}$




$$\overset{(5.21)}{=} \mu_0 \int_F \underline{j} \cdot d\underline{f}$$

$$\longrightarrow \boxed{\oint_{\partial F} \underline{B} \cdot d\underline{r} = \mu_0 I} \quad (5.24)$$

Zirkulation entlang $\partial F$

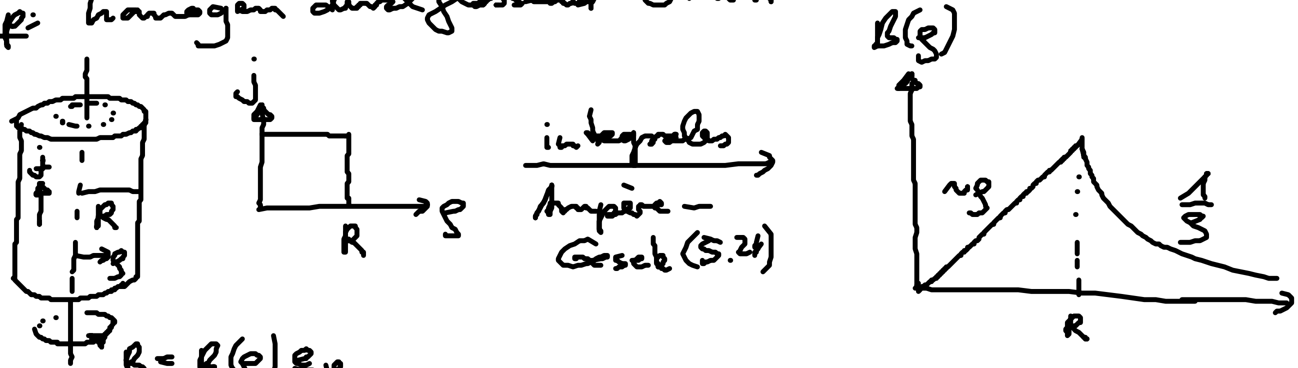
$= \mu_0 \times$ Strom durch F!

NB: unabh. von F bei gleichem $\partial F$!

· Poissonsche Gleichung für $\underline{A}$:

$$(5.21a) \longrightarrow \boxed{\begin{aligned} &\nabla^2 \underline{A} = -\mu_0 \underline{j} && (5.25) \\ &\text{mit } \underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{j}(\underline{r}')}{|\underline{r} - \underline{r}'|}\, d^3r' && (5.19) \end{aligned}}$$

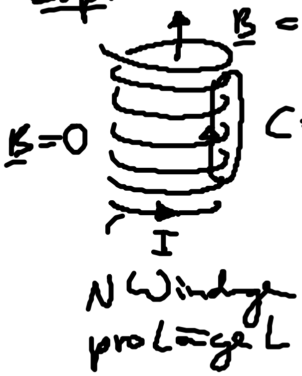
• **Bsp:** homogen durchflossener Draht




$$\underline{B} = B(\rho)\,\underline{e}_\varphi$$

Beweis: Übungen

• **Bsp:** unendlich lange Spule




(1) $$\oint \underline{B}\cdot d\underline{r} = L\,B_0 \overset{\text{Strom durch } F}{=} \mu_0 N I$$

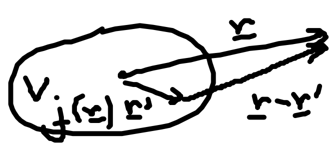
$$\Longleftrightarrow \quad B_0 = \mu_0 \frac{N}{L} I \qquad (5.26)$$

(2) $\underline{A} = A(\rho)\,\underline{e}_\varphi$ ... zirkuliert um $\underline{B}$!

Beweis: s. Übungen

## 5.4 Kleine Stromverteilungen: der magnet. Dipol

• Motivation: Lokalisierte Stromverteilungen




$\underline{A}(\underline{r})$? $\underline{B}(\underline{r})$? $\longleftrightarrow$ magn. Dipol: Definition & Eigenschaften

$$\text{statisch} \overset{(5.6)}{\Longleftrightarrow} \operatorname{div}\underline{j} = 0$$

### 5.4.1. Felder kleiner Stromverteilungen

• Entwickle nach Momenten der Stromverteilungen:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{j}(\underline{r}')}{|\underline{r}-\underline{r}'|}\,d^3r' \qquad (5.19)$$

$$\text{mit } \frac{1}{|\underline{r}-\underline{r}'|} \underset{\text{Taylor um } \underline{r}}{=} \frac{1}{r} + \underbrace{\frac{\underline{r}\cdot\underline{r}'}{r^3}}_{-\underline{r}'\cdot\nabla\frac{1}{r}} + \dots \qquad (\text{s. Kap. } 3.6)$$

$$\longrightarrow \underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int \underline{j}(\underline{r}')\, d^3r' + \frac{\underline{r}}{r^3}\cdot\int \underline{r}'\,\underline{j}(\underline{r}')\, d^3r' + \ldots\right] \quad (5.27)$$

(1) Hilfsatz:

$$\boxed{\begin{array}{l}\text{Für } \operatorname{div}\underline{j}=0 \text{ und beliebige } g, f \\ \text{gilt } \int \underline{j}\cdot\underline{\nabla}(gf)\, d^3r = 0\end{array}} \quad (5.28)$$

Beweis: $\int_V \underline{\nabla}\cdot(gf\underline{j})\, d^3r \overset{\text{Gauß}}{=} \int_{\partial V} gf\,\underline{j}\cdot d\underline{f} = 0$

$\underline{j}=0$, für $\partial V$ außerhalb von $\underline{j}\neq 0$

$\underline{j}\neq 0$ $\quad \underline{j}=0$ $\quad \underline{V}$ $\quad \partial V$

$$\overset{\text{Produkt-}}{\underset{\text{regel}}{=}} \int \underline{j}\cdot\underline{\nabla}(gf)\, d^3r + \int gf\, \underbrace{\underline{\nabla}\cdot\underline{j}}_{=0}\, d^3r \quad \text{qed}$$

(2) Es gilt: $\boxed{\int \underline{j}(\underline{r})\, d^3r = 0}$ (5.29)

... „es fließt nichts aus V heraus"

$\longrightarrow$ keine magnet. Monopole!

Beweis. (5.28) mit $g=1$, $f=x_i$

$$\longrightarrow \int j_k \underbrace{\nabla_k(x_i)}_{\delta_{ki}}\, d^3r = \int j_i\, d^3r = 0 \quad \text{qed}$$

(3) magnetisches Dipolmoment: (einer Stromverteilung)

Berechne 2. Term in (5.27):

$$\underline{r}\cdot\int \underline{r}'\, j_k(\underline{r}')\, d^3r' = x_i \int \underbrace{x_i' j_k(\underline{r}')}_{\text{Tensor 2. Stufe}}\, d^3r'$$

zerlege: $T_{ik} = \underbrace{\frac{1}{2}(T_{ik}+T_{ki})}_{\text{symmetrisch}} + \underbrace{\frac{1}{2}(T_{ik}-T_{ki})}_{\substack{\text{antisymmetrisch} \\ \hat{=}\text{ axialer Vektor}}}$

$$= x_i \int \Big[\underbrace{\tfrac{1}{2}(x_i' j_k + x_k' j_i)}_{(i)=0!} + \tfrac{1}{2}(x_i' j_k - x_k' j_i)\Big]\, d^3r'$$

(i): verwende $g = x_i'$, $f = x_k'$ in Gl. (5.28)

$$\longrightarrow \int j_l \nabla_l' (x_i' x_k') \, d^3r' = 0$$

$$= \int (j_i x_k' + j_k x_i') \, d^3r' = 0 \quad \text{qed}$$

zu (ii): $$= \frac{1}{2} \int \left[ (\underline{r} \cdot \underline{r}') \underline{j} - (\underline{r} \cdot \underline{j}) \underline{r}' \right] d^3r' = \frac{1}{2} \int (\underline{r}' \times \underline{j}) \times \underline{r} \, d^3r' \qquad (5.30)$$

$\longrightarrow$ Führe ein:

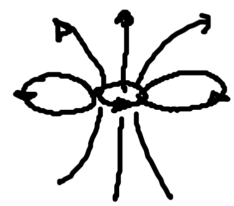
$$\boxed{\underline{m} = \frac{1}{2} \int \underline{r}' \times \underline{j}(\underline{r}') \, d^3r'} \qquad (5.31)$$

... magnet. Dipolmoment

$$\xrightarrow[\& (5.31)]{\text{Gl. } (5.27)} \quad \boxed{\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \underline{r}}{r^3}} \qquad (5.32)$$

... Vektorpotential des magnet. Dipols

$$\xrightarrow[\underline{r} = r \hat{\underline{r}}]{} \quad \boxed{\begin{aligned} \underline{B}(\underline{r}) &= \text{rot} \underline{A} \overset{\text{o.B.}}{=} - \underline{\nabla} \frac{\underline{m} \cdot \underline{r}}{r^3} \\ &= \frac{\mu_0}{4\pi} \frac{3(\underline{m} \cdot \hat{\underline{r}}) \hat{\underline{r}} - \underline{m}}{r^3} \end{aligned}} \qquad (5.33)$$



... magnet. Induktion von $\underline{m}$

(in völliger Analogie zum elektr. Dipol)