Introductory lecture Course of GRK 1558:

Delayed Complex Systems

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1. Introduction

Complex system: nonlin. dynamical system

- interacting subunits
- collective dynamics
- far from thermodyn. equilibrium
- bifurcation: change of stability
  branching of solutions

time evolution described by

- iterated maps (discrete time t_n):

$$x_{n+1} = f(x_n) \quad \text{nonlin. fct.}$$

- ordinary diff. eqs. (ODE)

$$\dot{x}(t) = f(x(t)) \qquad x \in \mathbb{R}^n$$

(coupled nonlin. ODEs $\dot{x}_i = f_i(x_1, \dots, x_n)$)

- partial differential eqs (PDE)

$$\dot{x}(t) = f(x) + D\Delta x$$

(e.g. nonlin. react.-diff. eq.)

Δ Laplacian

D diffusion constant

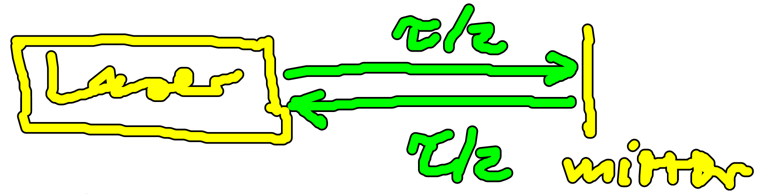
Delay in complex nonlinear systems

delay time τ, eg. $\dot{x}(t) = f(x(t)) + g(x(t-\tau))$

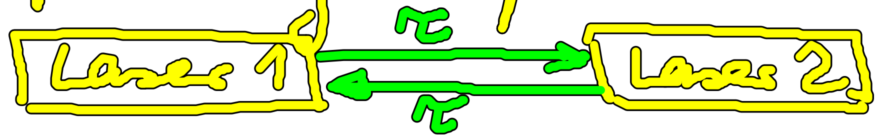
<u>delay diff. eq.</u> (DDE)

<u>delay is ubiquitous</u> in nonlinear systems:

- mech. systems: balancing segway
- electronic systems: latency due to signal processing (nanostructures)
- optical systems: signal transmission times light propagation
  - laser with optical feedback

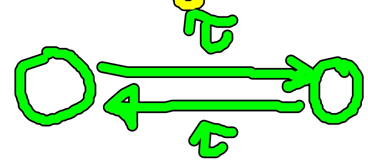


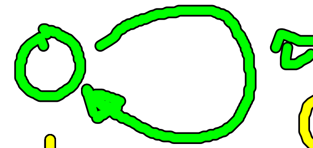
  - optically coupled lasers



- biological systems: cell cycle time
  biological clocks

– neural networks: delayed coupling





biochem. (neurotransmitt.)

<u>Why is delay interesting in dynamics?</u>

- Delay increases the dimension of an ODE, e.g. $\dot{x}(t) = -a x(t) + b x(t-\tau)$ (linear ODE of first order, but delay τ generates <u>infinitely</u> many eigenmodes)
- Simple eqs. produce very <u>complex</u> behavior:

- delay-induced bifurcations, instab.
- delay-induced multistability
- stabilization of unstable periodic or stationary states
- chaos control (suppression of chaos)

Lit.: Just, Pelster, Schanz, Schöll (eds):
Delayed complex system
(Theme Issue of Phil. Trans. Roy. Soc. A 368 (2010))

E. Schöll, H.G. Schuster (ed.): Handbook of Chaos Control (Wiley 2008)

T. Erneux: Applied Delay Diff. Eqs. (Springer 2009)

F. Atay (ed.): Complex time-delay syst. (Springer 2010)

Vorlesung „Nichtlineare Dynamik u. Kontrolle"
E. Schöll, WS 2012/13

2. Linear stability analysis

2.1 Linear delay diff. eq.

simplest model $\dot{x} = -a x(t) + b x(t-\tau)$ $a, b \in \mathbb{R}$

initial cond. $x(t) = \phi(t)$ $\boxed{-\tau \le t \le 0}$

history fct.

fixed point: $x^* = 0$

small perturbation of fixed point: $x(t) \sim e^{\Lambda t}$

a) $b = 0$; $\Lambda e^{\Lambda t} = -a e^{\Lambda t} \Leftrightarrow \Lambda = -a$

$a > 0 \Rightarrow \Lambda < 0$, $x(t) \xrightarrow{t \to \infty} 0$ fixed pt. stable

$a < 0 \Rightarrow \Lambda > 0$, $x(t) \to \infty$ " unstable

b) $b \neq 0$; $\Lambda e^{\Lambda t} = -a e^{\Lambda t} + b e^{\Lambda t} e^{-\Lambda \tau}$

$$\boxed{\Lambda = -a + b e^{-\Lambda \tau}}$$ transcendental eq. $\Lambda \in \mathbb{C}$

Solution for Λ: $\overbrace{(\Lambda + a)\tau}^{z} = b\tau e^{-\Lambda\tau}$

$$\underbrace{(\Lambda + a)\tau}_{z}$$

$$z e^{z} = b\tau e^{a\tau}$$

inverse fct. of $z e^{z} = y$

$z = W_\ell(y)$ Lambert fct.

(multivalued, $\ell \in \mathbb{Z}$

(cf. $e^{z} = y \Leftrightarrow z = \ln y$)

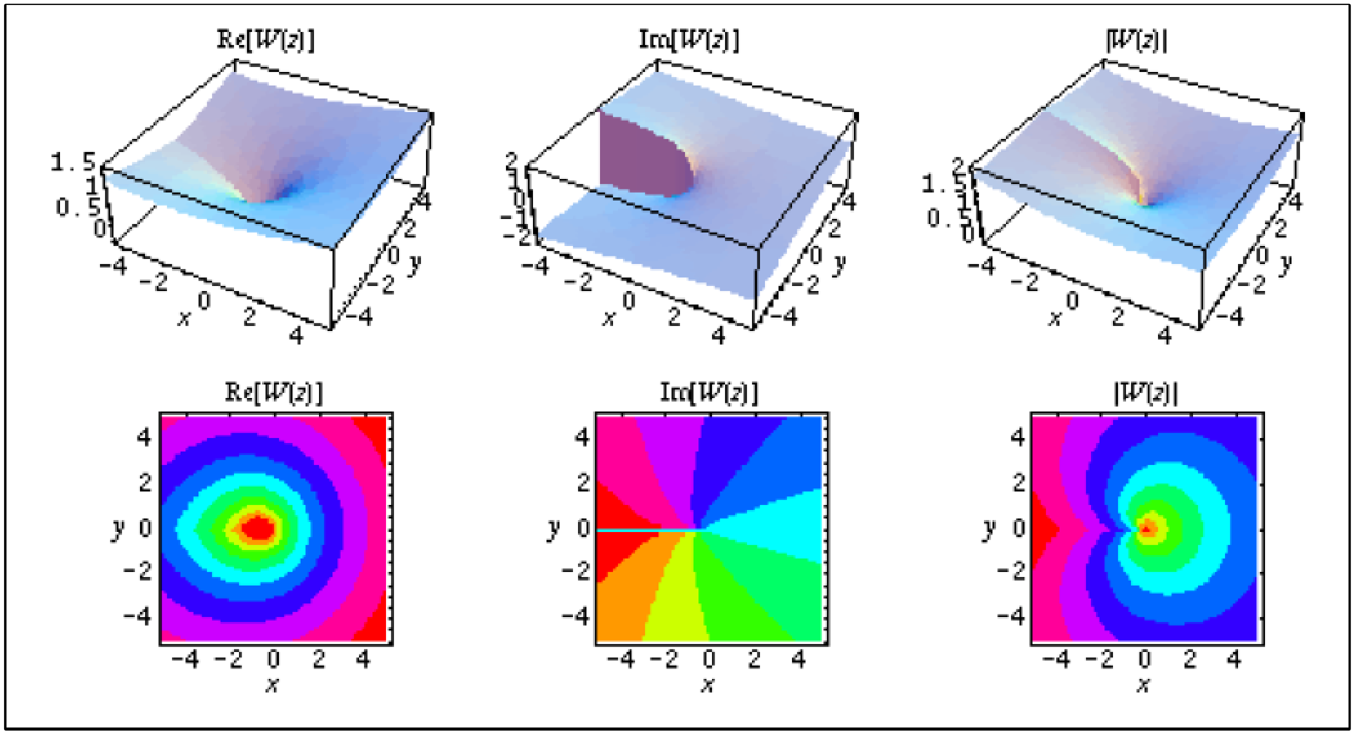
$$\Lambda_\ell = -a + \frac{1}{\tau} W_\ell\left(b\tau e^{a\tau}\right)$$

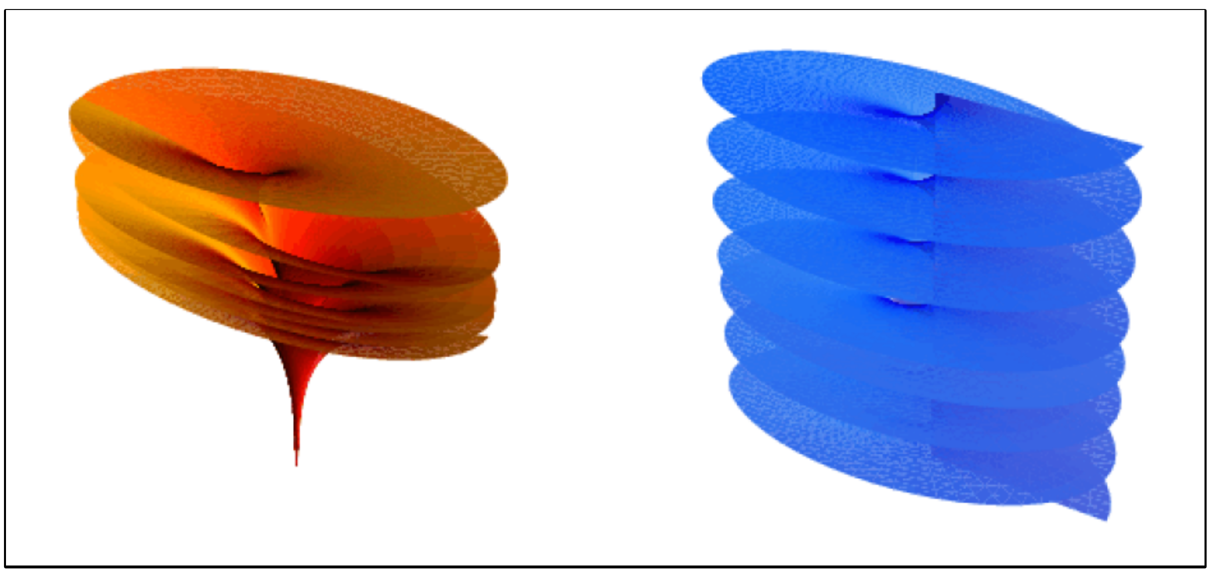
main branch: $W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n \quad \left(|z| < \frac{1}{e}\right)$

asymptotic expansion for $z \to 0$ and $z \to \infty$ ($\ell \neq 0$):

$$W_\ell(z) \approx \ln z + 2\pi i \ell - \ln(\ln z + 2\pi i \ell)$$

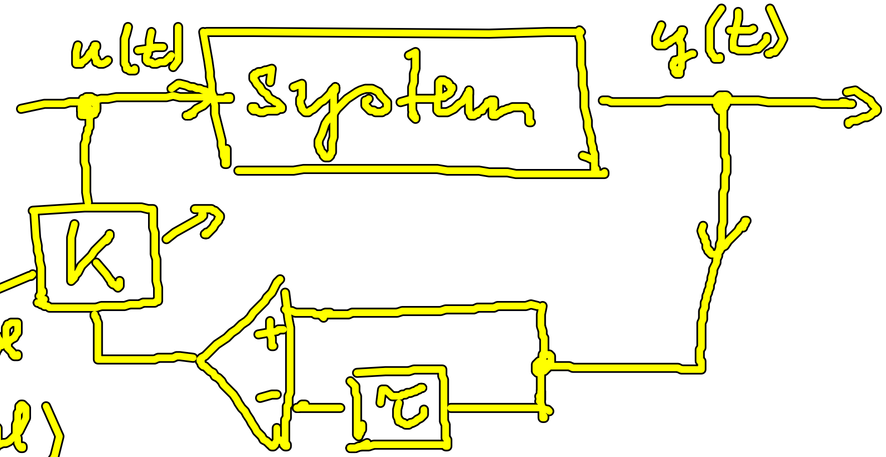
$$\tau \to 0 \; (z \to 0): \quad W_\ell(z) \approx \underbrace{\ln z}_{-\infty} + 2\pi i \ell + \text{higher order}$$





2.1 Stabilization of unstable fixed point by time-delayed feedback control (TDFC)

Pyragas (1992)



closed-loop control
(feedback control)

$y(t)$ output var.

$u(t) = K\,[y(t) - y(t-\tau)]$ control signal (input var.)

delay time τ

feedback strength (gain) K

- noninvasive

General form of a 2-var. dyn. system

fixed point x^*: $0 \overset{!}{=} \dot{\underline{x}} = \underline{f}(\underline{x}^*)$ $\quad \underline{x} \in \mathbb{R}^2$

linear. around $\underline{x}^*$ for small perturbations

$\underline{x}(t) = \underline{x}^* + \delta \underline{x}(t)$: $\quad \delta \dot{\underline{x}} = (Df)_* \, \delta \underline{x}$ $\quad$ Jacobian matrix

$(Df)_* \equiv A$

Sol. $\delta \underline{x} \sim e^{\Lambda t}$: $\quad 0 = \det(A - \Lambda \mathbb{1})$

$$= \det \begin{pmatrix} A_{11} - \Lambda & A_{12} \\ A_{21} & A_{22} - \Lambda \end{pmatrix}$$

$$\Rightarrow \Lambda = \frac{\mathrm{tr} A \pm \sqrt{(\mathrm{tr} A)^2 - 4 \det A}}{2}$$

<u>normal form of unstable focus</u>

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues $\Lambda = \lambda \pm i\omega$

$(\lambda > 0)$

complex form: $\dot{z} = (\lambda \pm i\omega) z$; $z = x + iy \in \mathbb{C}$

<u>with time-delayed feedback:</u>

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - K \begin{pmatrix} x(t) - x(t-\tau) \\ y(t) - y(t-\tau) \end{pmatrix}$$

"diagonal coupling"

Ansatz $x(t) = e^{\Lambda t}$

char. eq.: $0 = \det\left[\begin{pmatrix} \lambda - \Lambda & \omega \\ -\omega & \lambda - \Lambda \end{pmatrix} - K \begin{pmatrix} 1 - e^{-\Lambda\tau} & 0 \\ 0 & 1 - e^{-\Lambda\tau} \end{pmatrix}\right]$

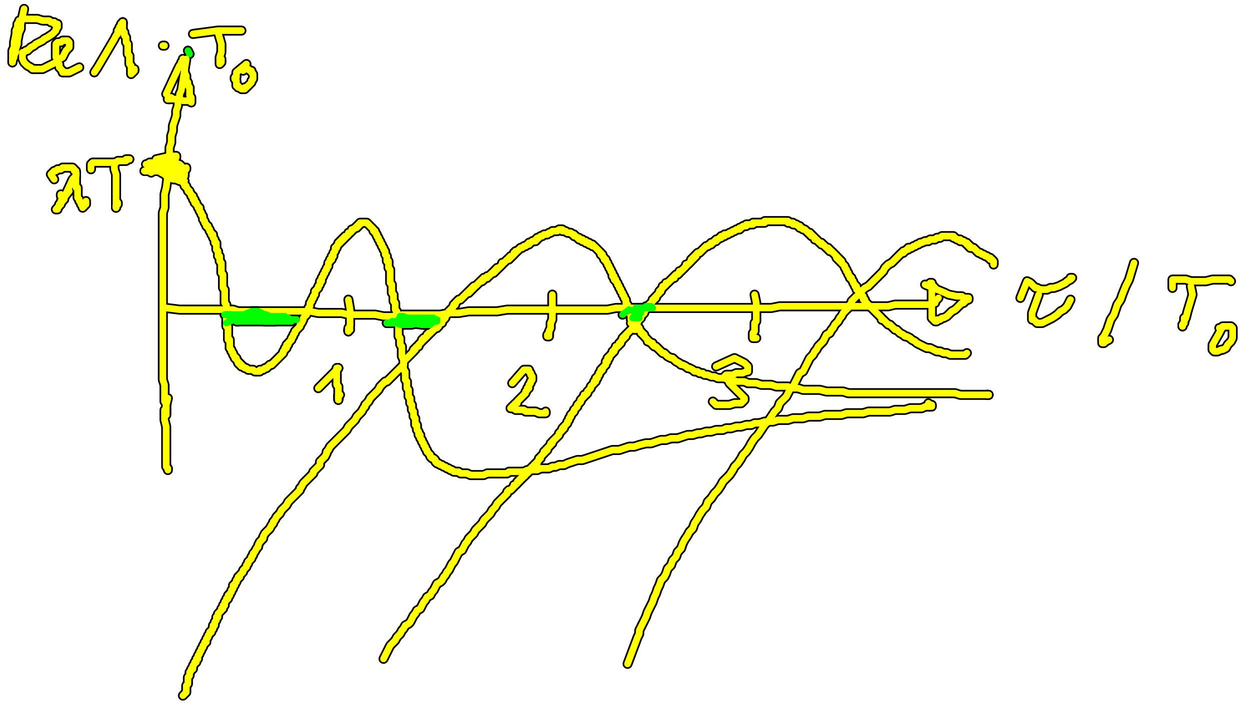
$$= \left[\Lambda + K(1 - e^{-\Lambda\tau}) - \lambda\right]^2 + \omega^2$$

$$\boxed{\Lambda + K(1 - e^{-\Lambda\tau}) = \lambda \pm i\omega}$$

solution by Lambert fct.: $\underbrace{(\Lambda + K - (\lambda \pm i\omega))\tau}_{z} = K\tau e^{-\Lambda\tau}$

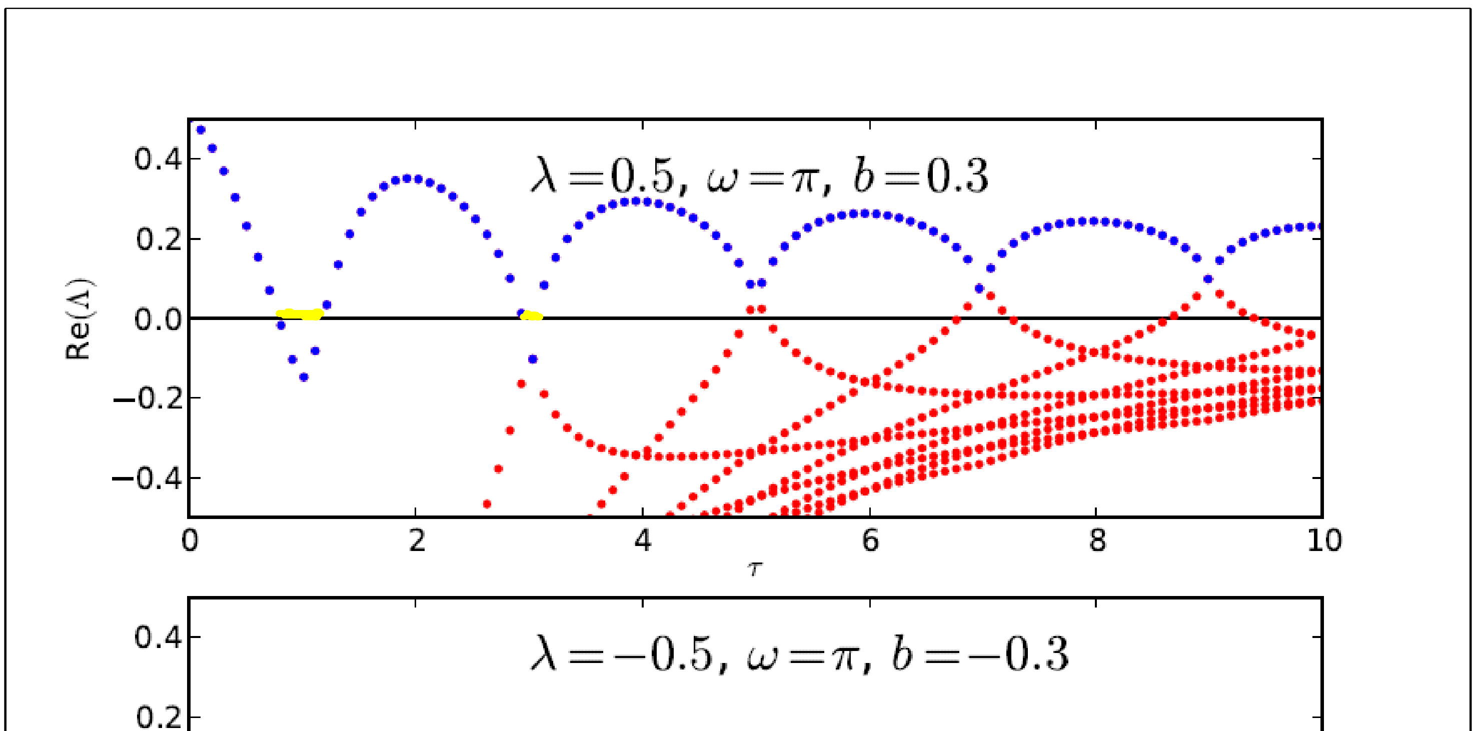
$$\Lambda\tau = W\left(K\tau e^{-(\lambda \pm i\omega)\tau + K\tau}\right) + (\lambda \pm i\omega)\tau - K\tau$$

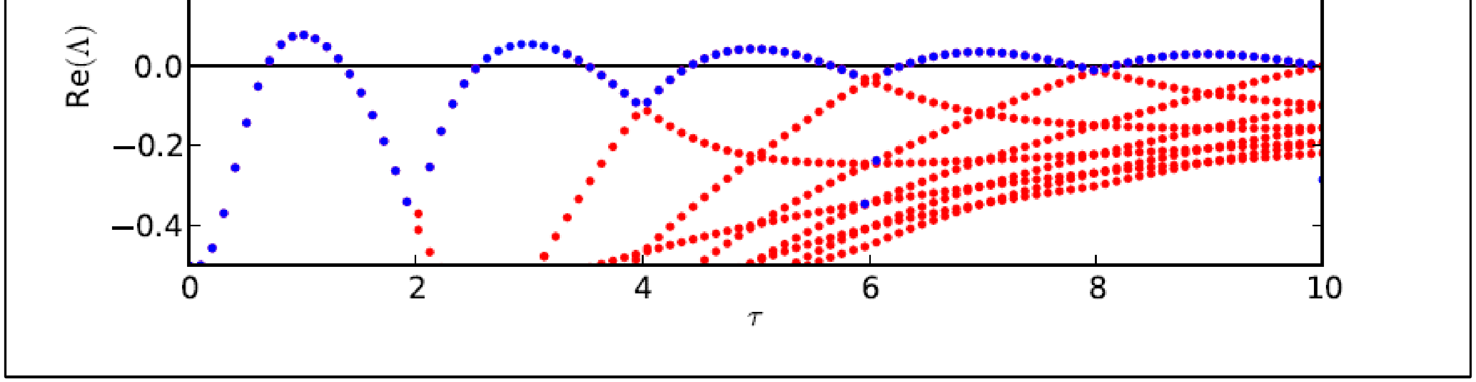
natural time-scale : $T_0 = \frac{2\pi}{\omega}$ (osc. period without delay)



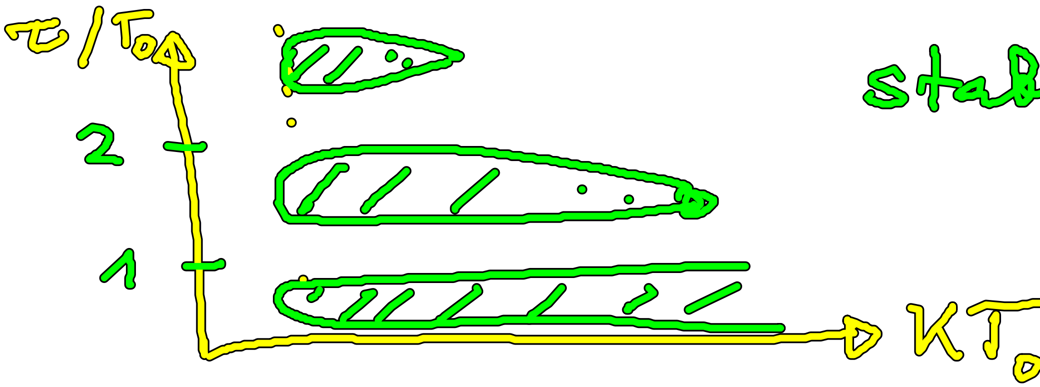
Stabilization for suitable τ and K ?

$\tau \approx \frac{T_0}{2}, \frac{3T_0}{2}, \ldots$





Stab. boundary.



stabilization tongues

Extensions

- multiple-time feedback control
- latency effects
- phase-dependent coupling
- asymptotic scaling for large

Stabilization of unstable periodic orbits

Chaos control

Spatio-temporal patterns can be stabilized