Complex Ginzburg-Landau Equation
Lecture II
Outline

- Plane Waves
- Absolute and Convective Instability
- Multiplicity of Solutions, Sinks and Sources
- Topological Defects
- Bulk and Core Instabilities
- Amplitude and Phase Chaos
- Spatio-Temporal Intermittency
Complex Ginzburg-Landau Equation

\[
\frac{\partial A}{\partial t} = A + (1 + ib)\Delta A - (1 + ic) |A|^2 A
\]

- \(A(x,y,t)\) – complex amplitude
- \(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\) Laplace operator
- \(b\) – linear dispersion
- \(c\) – nonlinear dispersion
Generic Properties of CGLE

\[ \frac{\partial A}{\partial t} = A + (1 + ib)\Delta A - (1 + ic) |A|^2 A \]

- Translation Invariance: \( r \rightarrow r + \text{const} \)
- Isotropic: angle \( \theta \rightarrow \theta + \text{const} \)
- Gauge Invariant: \( A \rightarrow Ae^{i\varphi}, \varphi = \text{const} \)
- Inversion in param space: \( (b, c, A) \rightarrow (-b, -c, A^*) \)
- Hidden symmetries (inherited from the NSE)
Select Solutions of the CGLE

- Plane Waves Solutions: 1D
- Spirals: 2D
- Vortex Filaments: 3D
Plane-wave solutions

\[ A = A_0 \exp[-i\Omega t + iqx + i\varphi_0] \]

- \( A_0 \) – amplitude
- \( \Omega \) – frequency
- \( \varphi_0 = const \) – arbitrary phase

\[ V_g = \frac{\partial \Omega}{\partial q} \neq 0 \] – group velocity

\[ V_{ph} = \frac{\Omega}{q} \neq V_g \] – phase velocity

For \( b=c=0 \) (real GLE) \( V_g = \omega = 0 \)
Plane-wave solutions

\[ A = A_0 \exp[-i\Omega t + iqx + i\phi_0] \]

\[-i\Omega A_0 = A_0 - (1 + ib)q^2 A_0 - (1 + ic)A_0^3\]

\[ \Omega = bq^2 + cA_0^2 \quad \text{imaginary part} \]
\[ 0 = 1 - q^2 - A_0^2 \quad \text{real part} \]
Plane-wave solutions

\[ A_0 = \sqrt{1-q^2} \quad \text{– amplitude} \]
\[ \Omega = c - (c - b)q^2 \quad \text{– frequency} \]
\[ \varphi_0 = \text{const} \quad \text{– arbitrary phase} \]
\[ V_g = \frac{\partial \Omega}{\partial q} = -2(c - b)q \neq 0 \quad \text{– group velocity} \]
\[ V_{ph} = \frac{\Omega}{q} \neq V_g \quad \text{– phase velocity} \]

For \( b=c=0 \) (real GLE) \( V_g = \omega = 0 \)
Stability of Plane Waves

- We look for the solution in the form

\[ A = (A_0 + \xi) \exp[-i\Omega t + iqx], \xi - \text{small perturbation} \]

- Linearized CGLE

\[ \frac{\partial \xi}{\partial t} = (1 + ib)(\Delta \xi + 2iq\partial_x \xi) - (1 + ic)|A_0|^2(\xi + \xi^*) \]
Linear Eqs: Re & Im Parts

• Rewrite solution in the form: $\xi = u + iw$, $u = \text{Re } \xi$, $w = \text{Im } \xi$

$$\frac{\partial u}{\partial t} = \Delta (u - bw) - 2q \partial_x (bu + w) - 2 |A_0|^2 u$$

$$\frac{\partial w}{\partial t} = \Delta (bu + w) + 2q \partial_x (u - bw) - 2c |A_0|^2 u$$

• Perturbative solution

$$\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} U_0 \\ W_0 \end{pmatrix} \exp[\lambda t + ikx]$$

$k$ – modulation wavenumber, $U_0, W_0 = \text{const}$
Stability Conditions

- Linear growth rate *(check at home!!!)*

\[ \lambda^2 + 2(A_0^2 + 2ibkq + k^2)\lambda + (1 + b^2)(k^4 - 4(qk)^2) + 2A_0^2 \left( (1 + bc)k^2 + 2i(bc)qk \right) = 0 \]

- Long-wave expansion

*(Home work: Calculate \( \Omega_g, D_4 \), use Mathematica)*

\[ \lambda = -iV_g k - D_2 k^2 + i\Omega_g k^3 - D_4 k^4 + O(k^5) \]

\[ V_g = 2(b - c)q - \text{group velocity} \]

\[ D_2 = 1 + bc - \frac{2(1 + c^2)q^2}{1 - q^2} - \text{phase diffusion} \]
Eckhaus and Benjamin-Feir Limits

- Onset of the Instability $D_2<0$

$$q^2 < q_E^2 = \frac{1 + bc}{3 + 2c^2 + bc} - \text{Eckhaus Criterion}$$

- Benjamin-Feir-Newell Criterion $q_E=0$

$$1 + bc = 0$$
Mysterious Restabilization

- Waves emitted by the spiral deep in unstable regions restabilize after some time???
If upon a cage of an elephant you will see a sign reading “buffalo”, do not believe your eyes!
Convective vs Absolute Instability

- For $V_g = 2(b-c)q \neq 0$ Eckhaus criterion is only a test for convective instability

  - Convective Instability: perturbations grow in moving frame and decays in lab frame
  - Absolute Instability: perturbations grow in lab frame
Criterion for Absolute Instability

- Evolution of localized perturbation
  \( S_0(k) \)-spectrum of init conditions
- For \( t \to \infty \) the integral is dominated by largest saddle point \( k_0 \)

\[
S(x, t) \sim \int_{-\infty}^{\infty} S_0(k) \exp[ikx + \lambda(k)t]dk
\]
Evaluation of Integral (saddle point method)

Expand growthrate in complex plane $k = k_0 + \kappa$

$$\lambda(k) = \lambda(k_0) + \lambda'(k_0)\kappa + \frac{1}{2} \lambda''(k_0)\kappa^2 + ...$$

$$S(x, t) \sim \int_{-\infty}^{\infty} S_0(k) \exp[ikx + \lambda(k)t] dk \approx \text{oscillate}$$

$$\exp[ik_0x + \lambda(k_0)t] \int_{-\infty}^{\infty} S_0(k_0) \exp[i\kappa x + \lambda'(k_0)\kappa t + \frac{1}{2} \lambda''(k_0)\kappa^2 t +...] dk \sim$$

$$S_0(k_0) \exp[ik_0 x + \lambda(k_0)t] \int_{-\infty}^{\infty} \exp[i\kappa x + \frac{1}{2} \lambda''(k_0)\kappa^2 t] dk =$$

$$\sim S_0(k_0) \exp[ik_0 x + \lambda(k_0)t] \exp[x^2 / (2\lambda''(k_0)t)]$$

$$\kappa = k - k_0$$

$$\text{Re}[\lambda(k_0)] = 0 \text{ at } \frac{\partial \lambda(k_0)}{\partial k} = 0 \quad \text{Stability limit}$$
Convective vs Absolute Instability

**Convective Instability Condition**

\[ \text{Re}[\lambda(k_0)] > 0 \text{ with } \text{Im}[\lambda(k_0)] \neq 0 \]

**Absolute Instability Condition**

\[ \text{Re}[\lambda(k_0)] > 0 \text{ with } \frac{\partial \lambda(k_0)}{\partial k} = 0 \]
Collision of waves

• In linear wave equation waves do not interact – superposition
• In the CGLE waves collide and annihilate
Collision of waves

- Consider 1D CGLE
- For simplicity $b=0$, $c<<1$

\[
\frac{\partial A}{\partial t} = A + \frac{\partial^2 A}{\partial x^2} - (1 + ic) |A|^2 A
\]
Collision of waves

\[ A = I \exp(i\varphi) \]

\[
\frac{\partial I}{\partial t} = I - I^3 + \frac{\partial^2 I}{\partial x^2} - \left( \frac{\partial \varphi}{\partial x} \right)^2 I
\]

\[
I \frac{\partial \varphi}{\partial t} = I \frac{\partial^2 \varphi}{\partial x^2} + 2 \left( \frac{\partial \varphi}{\partial x} \frac{\partial I}{\partial x} \right) - cI^3
\]

- Adiabatic approximation: neglect derivatives of \( I \)

\[
I - I^3 - \left( \frac{\partial \varphi}{\partial x} \right)^2 I = 0 \rightarrow I^2 = 1 - \left( \frac{\partial \varphi}{\partial x} \right)^2
\]
Burgers Equation

\[ \frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} - c + c \left( \frac{\partial \varphi}{\partial x} \right)^2 \]

Burgers equation can be integrated by the Hopf-Cole transformation to linear diffusion equation

\[ \varphi = \frac{1}{c} \log(W) \]

\[ \frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2} - c^2 W \]
Logarithmic Superposition

\[
\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2} - c^2 W
\]

\[W = C_1 \exp(cqx + c^2(q^2 - 1)t) + C_2 \exp(-cq x + c^2(q^2 - 1)t)\]

\[\varphi = \frac{1}{c} \log(W) = c(q^2 - 1)t + \frac{1}{c} \log \left( C_1 \exp(cqx) + C_2 \exp(-cq x) \right)\]

Assume \(C_1 \exp(cqx) \gg C_2 \exp(-cq x)\)

\[\varphi = c(q^2 - 1)t + qx + \frac{C_2}{cC_1} \exp(-2cq x)\]

*Opposite wave decays exponentially*

*Screening length \(\sim 1/2cq\)*
Wavenumber Selection

• Planes waves form a continuous family

\[ A = A_0 \exp[-i\Omega t + iqx], \quad 0 \leq |q| \leq 1 \]

\[ A_0 = \sqrt{1 - q^2} \] – amplitude

\[ \Omega = c - (c - b)q^2 \] – frequency

• What wavenumber to take?

• Wavenumber selected by boundaries, inhomogeneities or certain coherent structures (spirals, vortices, defects etc)
(Simple) Coherent Structures

- **Source:** group velocities directed outward of the defect sources dominate surrounding dynamics sources come in discrete families source selects the wavenumber of emitted waves
- **Sink (shock):** are results of collisions of incoming waves shocks are determined by boundary conditions or other sources shocks come in continuous families and don’t have specific structure
- **Phason:** group velocity does not change the sign bound state of shock and source
- **Periodic trajectories:** modulated traveling waves

*Classification of coherent structures from counting arguments van Saarloos & Hohenberg, Physics D, 1992*
Coherent structures connect two plane wave solutions \((P, R)\)

**Stationary solution in moving frame**

\[
A(x, y) = A_0(x - Vt)e^{i\omega t} = a(x - Vt)e^{-i\omega t + i\psi(x - Vt)}
\]
Counting eigenvalues

- Plane waves are fixed points of this system
- Homoclinic/ heteroclinic trajectories-coherent structures or domain walls
- We need to calculate unstable eigenvalues at each point (total $N_P$)
- There is $N_P - 1$ parameters charactering the flow (due to linearity)
- With $V$ and $\omega$ total $N_P + 1$ parameters
- Solution has to come orthogonal to unstable direction of other point (total $N_R$)
- Multiplicity of trajectories: $n = N_P + 1 - N_R$
- $n \geq 1$ – continuum family, $n = 0$ – discrete family, $n < 0$ – no structures
Counting Arguments: CGLE example

Stationary solution in moving frame

\[ A(x, y) = A_0 (x - Vt)e^{i\omega t} = a(x - Vt)e^{-i\omega t + i\psi (x - Vt)} \]

In co-moving frame \( \xi = x - Vt \)  3d order ODE for \( a, \frac{da}{d\xi}, \) and \( s = \frac{d\psi}{d\xi} \)

\[
\begin{align*}
\partial_{\xi}^2 a + (1 - s^2 - a^2) a + V \partial_{\xi} a &= 0 \\
n a \partial_{\xi} s + 2s \partial_{\xi} a + a\omega - ca^3 + Vas &= 0
\end{align*}
\]
Fixed points

\[ \partial^2_{\xi} a + (1 - s^2 - a^2) a + V \partial_{\xi} a = 0 \]

\[ a \partial_{\xi} s + 2 s \partial_{\xi} a + a \omega - c a^3 + V a s = 0 \]

Fixed Points

\[ s = s_0 = \text{const}, \; |s_0| \leq 1, \; a_0 = \sqrt{1 - s_0^2}, \; \omega = c(1 - s_0^2) - V s_0 \]

Group Velocity

\[ V_g = \partial \omega / \partial s = -2c s_0 - V \]
Linearized system

$$\frac{\partial^2}{\partial \xi^2} a + (1 - s^2 - a^2) a + V \frac{\partial}{\partial \xi} a = 0$$

$$\frac{\partial}{\partial \xi} s + \frac{2s}{a} \frac{\partial}{\partial \xi} a + \omega - ca^2 + Vs = 0$$

$$a = A + a_0, s = S + s_0$$

$$\frac{\partial^2}{\partial \xi^2} A - 2(s_0 S + a_0 A) a_0 + V \frac{\partial}{\partial \xi} A = 0$$

$$\frac{\partial}{\partial \xi} S + \frac{2s_0}{a_0} \frac{\partial}{\partial \xi} A - 2ca_0 A + VS = 0$$

$$S, A \sim \exp[\lambda \xi]$$

$$\begin{pmatrix}
\lambda^2 + \lambda V - 2a_0^2, & -2a_0 s_0 \\
\frac{2s_0}{a_0} \lambda - 2ca_0, & \lambda + V
\end{pmatrix} = 0$$

$$(\lambda^2 + \lambda V - 2a_0^2)(\lambda + V) + 2a_0 s_0 \left( \frac{2s_0}{a_0} \lambda - 2ca_0 \right) = 0$$

Eigen Values (small $V, c, s$)

$$\lambda_{1,2} \approx \pm \sqrt{2} a_0, \lambda_3 \approx -2cs_0 - V = V_g$$

Sign is determined by the direction of group velocity
Keep Counting …

\[ \lambda_3 = V_g \]

\[ N_p = 1 \quad \text{source} \quad N_R = 2 \]

\[ n = N_p + 1 - N_R = 0: \quad \text{Discrete family of sources} \]

\[ N_p = 2 \quad \text{Sink (shock)} \quad N_R = 1 \]

\[ n = N_p + 1 - N_R = 2: \quad \text{2-parametric family of sinks} \]
Discrete Family of Sources!!!

- Are the calculations correct?
- Yes
- Is the answer correct?
- NO
- What to do: remember Kozma Prutkov!
- Hidden symmetry of the 1d CGLE inherited from the Nonlinear Schrödinger Eq.
Nozaki-Bekki Holes

- Exact solution – continuum family of moving sources

\[ A = \left[ B \partial_\xi \varphi(\kappa \xi) + aV \right] \exp \left[ i\varphi(\kappa \xi) + i\alpha V - i\omega t \right] \]

\[ \xi = x - Vt, \varphi(\kappa \xi) = \frac{1}{\kappa} \log \cosh(\kappa \xi) \]

\[ V = (b - c)(q_1 + q_2) \]

- NB holes exist “for no good reason”,
- they are structurally unstable

*Popp, Stiller, Aranson, Kramer, Physica D 1995*
Destruction of NB holes and Intermittency

- NB holes form continuous family (hidden symmetry, non-robust)
- Generic perturbations to the CGLE break the family
- Depending on perturbations holes either slow down or accelerate

\[
\frac{\partial A}{\partial t} = A + (1 + ib) \frac{\partial^2 A}{\partial x^2} - (1 + ic) |A|^2 A \\
+ d |A|^4 A, d \ll 1
\]

*Popp, Stiller, IA and L. Kramer, PRL 1995*
Spatio-Temporal Intermittency in 1D

Zigzagging holes (M van Hecke)

\[ b = -0.6 \]
\[ c = 1.4 \]
Topological Defects

- Zeros of $A = |A| e^{i\theta}$ result in topological singularity for phase $\theta = \text{arg}A$

$$n = \frac{1}{2\pi} \oint \nabla \theta dl \quad \text{topological charge, } n = \pm 1$$

- 2D – spirals (vortices), 3D – vortex lines or filaments
Active and Passive Defects

Active defect
(spiral)

Passive defect
(shock)

Shock lines
Why are the topological defects important

- Active TDs (sources) dominate surrounding dynamics
- Stability of TDs determine transition to spatio-temporal chaos
- TDs are “elementary excitations” of the medium. Evolution the CGLE can be formulated in terms of of the “gas” of interacting TDs
- TDs can coexist with turbulence and exhibit spatio-temporal intermittency in 1D, 2D and 3D (the states now would be called chimeras)
Spiral Solution to CGLE

\[ A = F(r) \exp[-i\omega t \pm i\theta + i\psi(r) + i\varphi_0] \]

\( r, \theta \) – polar coordinates, \( \varphi_0 = \text{const} \) – phase

\( F(r), \psi(r) \) – unique monotonous functions

\( \omega \) – rotation frequency
Spiral Solution to CGLE

\[ A = F(r) \exp[-i\omega t \pm i\theta + i\psi(r) + i\varphi_0] \]

\( F(r), \psi(r) \) – unique monotonous functions

\( \psi(r) \to kr \) for \( r \gg 1 \), \( k \) – asymptotic wavenumber

\( \psi(r) = \theta + kr \) – Archimedes Spiral

\( F(0) = 0, F(\infty) = \sqrt{1 - k^2} \)

\( \omega = c - (c - b)k^2 \) – rotation frequency

\[ k \sim \exp\left[-\frac{\pi}{2|c - b|}\right] \text{ for } c - b \to 0 \]
Interaction of Spiral Waves

- If well-separated, spiral waves are close to plane waves
- Counter-propagating plane waves annihilate and decay
- Interaction is exponentially screened
Interaction of Spiral Waves for $c << 1$

- Interaction of spiral is similar to collision of plane waves
- For $c << 1$ is described by the Burgers equation

$$\frac{\partial \varphi}{\partial t} = \nabla^2 \varphi - c + c(\nabla \varphi)^2$$

- Solution is logarithmic superposition
- Interaction decays as $\exp(-2c q R)$, $R$ distance between spiral cores

$$V_{\text{drift}} \sim \exp(-2c q R)$$
General Case: arbitrary $b$ & $c$

- Dispersion relation
  \[ A = A_0 e^{iqx - i\omega t} + \xi_+ \exp[\lambda t + ikx] + \xi_- \exp[\lambda^* t - ikx] \]

  \[ \lambda^2 + 2(A_0^2 + 2ibqk + k^2)\lambda + (1 + b^2)(k^4 - 4(qk)^4) + 2A_0^2 \left( (1 + bc)k^2 + 2i(b - c)qk \right) = 0 \]

- For stationary solutions ($\lambda=0$) take $p=ik$ gives

  \[ p(1 + b^2)(p^2 + 4q^2) = 2A_0^2 \left( (1 + bc)p - 2(b - c)q \right) = 0 \]

- $p=0$ – always a solution (due to translational invariance)
Monotonous and Oscillatory Ranges

- For stationary solutions ($\lambda=0$) take $p=ik$ gives

$$(1+b^2)(p^2+4q^2)p - 2A_0^2((1+bc)p - 2(b-c)q) = 0$$

- For small $(b-c)$ & $q$ we obtain

$$p \approx 2(b-c)q = V_q - \text{monotonic interaction}$$

- For larger $(b-c)$ we have two complex roots

-oscillatory range

$$\frac{c-b}{1+bc} > c_{cr} = 0.85$$
Interaction between two spirals

**Exponential decay of interaction**

- **Monotonic range** $c<0.845$ & $b=0$:
  - weak repulsion
  - irrespectively of charge
  - no symmetry breaking
- **Oscillatory range** $c>0.845$:
  - oscillatory interaction
  - vs distance
  - meta-stable bound states
  - symmetry breaking

- $v$ – velocity of the spiral core
- $\phi$ – phase of the spiral
- $X$ – distance to the shock line

**References**
- Biktashev, 1989 (small $c$)
Oscillatory Case: Asymptotic Method

- Spiral pair → spiral with the boundary
  \[ A = \left( F(r) + \xi \right) \exp[i \omega t + i \theta + i \psi(r) + i \phi_0(t)] \]
  \[ r \to r - r_0(t), \quad v = \dot{r}_0(t) - \text{spiral core velocity} \]
  \[ L \xi = v \nabla A_0 + i \phi_0 A_0 \]
  \[ \xi = C_1 r^\mu \exp[-pr + i \theta] + C_2 r^\mu \exp[pr + i \theta] \quad \text{for} \quad r \to \infty \]
  \[ \xi = c_m r^m, m = 0, m = \pm 2 \quad \text{for} \quad r \to 0 \]

- Constants \( C \) determined from the b.c
- \( v(t) \) and \( \phi_0(t) \) determined from the numerical orthogonalization at the core
Oscillatory Case

\[ \nu_n = \text{Im} \left( \frac{-k \sqrt{1 - k^2} e^{-pX}}{\delta C_y \sqrt{2\pi} pX} X^{-\mu} \right) \text{ / } \text{Im}(C_x \ / \ C_y) \]

\[ \nu_\tau = \text{Re} \left( \frac{-k \sqrt{1 - k^2} e^{-pX}}{\delta C_y \sqrt{2\pi} pX} X^{-\mu} \right) - \nu_n \text{ Re}(C_x \ / \ C_y) \]

\[ \varphi = \text{Im} \left( \frac{-k \sqrt{1 - k^2} e^{-pX}}{\delta C_{00} \sqrt{2\pi} pX} X^{-\mu} \right) \text{ / } \text{Im}(C_{10} \ / \ C_{00}) \]

- Complex \( p \) - multiple bound states
- Same charge - rotate, oppositely charged – drift
- Symmetry breaking of bound states

\[ C_{x,y}, C_{10,00}, \delta \text{ and (complex) } p, \mu \text{ determined numerically by Aranson, Kramer & Weber, PRE, 1993} \]
Oscillatory Case

\[ V_{\text{drift}} \sim \exp(-pR) \]

- Complex \( p \) - multiple bound states
- Only first bound state is observable
- Same charge - rotate,
- Oppositely charged – drift
Monotonic case: pair interaction

\[ r \to 2c \mid k \mid r, \quad t \to 4\sqrt{2\pi c^3 k^2}, \quad \zeta \to c \varphi \]

screening length \( l = \frac{2\pi}{k} \sim \exp\left(\frac{\pi}{2c}\right) \to \infty \) for \( c \to 0 \)

\[ v_n = -cB' \frac{e^{-X}}{\sqrt{X}} ; \quad v_r = m \frac{e^{-X}}{\sqrt{X}} ; \quad \zeta = \frac{1}{2c} \frac{e^{-X}}{\sqrt{X}} \]

\[ X = |r_j - r_k|/2 + (\zeta_j - \zeta_k) \]

equations for "complex" position \( z = x + iy \)

\[ z_j = (cB' + i m_k) \frac{r_j - r_k}{|r_j - r_k|} \frac{e^{-X}}{\sqrt{X}} \]

• “Oblique” interaction
  • Oppositely charged spirals repel and drift
  • Likely charged spirals repel and rotate
Examples of Bound States

**Oppositely charged-drift**

\[ c=1.5, b=0 \]

**Likely charged-rotate**
Symmetry Breaking

Big Brother always wins!!!
Final State after Symmetry Breaking

Spiral defect

sink
The Origin of Symmetry Breaking

- If the shock not in the middle

\[ V_{\text{shock}} = \frac{1}{2} \left( \frac{\omega_1}{k_1} + \frac{\omega_2}{k_2} \right) \]

- If the frequency \( \omega \) decreases with the distance to the shock, the symmetric state is unstable
• OR-oscillatory range (symmetry breaking)
• EI – Eckhaus instability for spirals
• BF – Benjamin-Fair limit
• AI- absolute instability for spirals and transition to chaos
Vortex Glass in Large System
Bulk and Core Instabilities of TD

- Active TDs select unique wavenumber and have a fixed core structure
- Bulk instability (BI) – emitted waves are unstable (continuum spectrum)
- Core instability (CI) : discrete localized mode are unstable
- Transitions to chaos can be related either BI or CI, or both of them
Examples of BI and CI

2D Spiral Intermittency
(BI+CI), c=-0.4, b=40

3D Vortex stretching
(CI) c=-0.03, b=50
Understanding Spiral Core Instability

• For $b \to \infty$ CGLE can be written as,

$$\varepsilon = \frac{1}{b}, \quad r \to \frac{r}{\sqrt{b}}$$

$$\frac{\partial A}{\partial t} = A + (\varepsilon + i) \Delta A - (1 + ic) |A|^2 A, \varepsilon \ll 1$$

$\varepsilon = 0$ there exists family of moving spirals with velocity $v$

$$A = F(r) \exp[i\omega't \pm i\theta + i\psi(r) - ir'v / 2]$$

$$r' = r + vt, \quad \omega' = \omega - |v|^2 / 4$$

• For $\varepsilon \neq 0$ the family breaks down and spirals accelerate
Asymptotic Method

\[ A_0 = F(r) \exp[i \omega t \pm i \theta + i \psi(r) - ir'v / 2] \]
\[ r' = r + vt, \quad \omega' = \omega - v^2 / 4 \]
\[ A = A_0 + w \]
\[ \hat{L}_W = H = \frac{ir\dot{v}}{2} A_0 - \varepsilon i v \nabla A_0 \]
\[ \hat{L}^\dagger \neq \hat{L}^* \]
Adjoint Eigenmodes
Asymptotic Method

- \( V \)-slowly varying function of time

\[
\frac{d}{dt} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \varepsilon \begin{pmatrix} K_{xx} & K_{xy} \\ -K_{xy} & K_{yy} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}
\]

\[
\frac{d\hat{V}}{dt} = \varepsilon \kappa \hat{V}
\]

\( \hat{V} = v_x + iv_y \) - "complex velocity"

\( \kappa = \kappa_{xx} + i\kappa_{xy}, \kappa_{xx} = \kappa_{yy} \) - "complex friction"

\( \kappa_{xx} > 0 \)
Core Instability Limit

- Acceleration – growth of localized mode at the core
- This mode can be found numerically for arbitrary $\varepsilon$ and analytically for small $\varepsilon$ (I.A. Weber and Lorenz Kramer, PRL 1994)
- No saturation of the instability
Simultaneous Core and Eckhaus Instabilities

Oscillations of number of defects
3D Vortices

\[ A = F(r) \exp[i \omega t \pm i \theta + i \psi(r) + i \eta z] \]
\[ \eta - \text{pitch} \]
\[ \eta = 0 - \text{straight vortex line} \]
\[ \eta \neq 0 - \text{twisted vortex line} \]
Vortex Ring
Collapse of Vortex Rings: \( b=0 \)

Cross-section of the ring

\[ \frac{\partial A}{\partial t} = A + \left( \frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial r^2} \right) + \frac{1}{r} \frac{\partial A}{\partial r} - (1 + ic) \left| A \right|^2 A \]

CGLE in cylindrical coordinates (no \( \varphi \) dependence)

\((R,0) - \text{position of the core}\)
Collapse of Vortex Rings: \( b=0 \)

**Slowly-drifting solution**

\[
A(r, z, t) = A(r - Vt, z)
\]

\[
-V \frac{\partial A}{\partial r} = A + \left( \frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial r^2} \right) + \frac{1}{R} \frac{\partial A}{\partial r} - (1 + ic) |A|^2 A
\]

\( V = \frac{dR}{dt} = -\frac{1}{R}, R^2 = R_0^2 - 2t \)

**Slightly curved vortex line straitens**

\[
V_n = -\kappa, \kappa - \text{local curvature}
\]
Collapse of Vortex Rings for $b \neq 0$

\[
\frac{dR}{dt} = -(1 + b^2) \frac{1}{R}
\]

\[
R^2 = R_0^2 - 2(1 + b^2)t
\]

Gabbay, Ott, Guzdar, PRL 1997
Another Famous Saying

If ever asked: What's more useful, the Sun or the Moon, respond: The Moon, of course. The Sun only shines during daytime, when it's light anyway, whereas the Moon shines at night.
Vortex Lines and Acceleration Instability

\[ \frac{d\hat{V}}{dt} = \kappa [\varepsilon \hat{V} - \chi] \]

\( \chi \) – curvature, \( \chi = \partial_s^2 \hat{r} \)

\( s \) – arclength

\( k \) – modulation wavenumber along z
3D vortices more unstable than spirals

- Instability occurs at finite wavenumber in z-direction
3D vortices more unstable than spirals

- Instability occurs at finite wavenumber in z-direction
Evolution of Vortex Ring

- Without CI vortex ring shrinks (Gabbay et al, 1997)
- With CI vortex ring breaks (Aranson & Bishop, 1997)
3D Core Instability and Symmetry Breaking

- Core Unstable Vortices in Oscillatory range show length oscillations
Amplitude (or defect) and Phase Chaos

- Phase Chaos:
  \[ |A| \approx \text{const} > 0, \text{ phase chaotic} \]
- Amplitude Chaos:
- amplitude \( A \) has zeroes

\[ c = 1.333, \ b = -1 \]

Chate et al, 1994
Burgers and Kuramoto-Sivashinsky Eqs.

- We write the CGLE in amp-phase form $A = I e^{i\varphi}$ (for simplicity $b=0$)

\[
\frac{\partial}{\partial t} I = \Delta I - I(\nabla \varphi)^2 + I - I^3
\]

\[
\frac{\partial}{\partial t} \varphi = \Delta \varphi + \frac{2}{I} \varphi \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} I + \omega - cI^2
\]

- For small gradients ($c$ is small)

\[
I^2 \approx 1 - (\nabla \varphi)^2
\]

\[
\frac{\partial}{\partial t} \varphi = \Delta \varphi + \omega - c + c(\nabla \varphi)^2 \quad \text{Burgers Eq.}
\]

- With Hopf-Cole transformation $\varphi = c^{-1} \log W$ obtain linear diff Eq

\[
\frac{\partial}{\partial t} W = \Delta W + (\omega - c)W
\]
General Solution to Burgers Equation

Logarithmic Superposition

\[
\varphi = \frac{1}{c} \log \left[ \sum C_k \exp \left( \lambda_k t + q_k x \right) \right]
\]

\[
\lambda_k = \omega - c - q_k^2
\]
Kuramoto-Sivashinsky Eq for larger $c$

- Linear growth rate for plane waves
  \[ \lambda = -iV_k k - D_2 k^2 + i\Omega_k k^3 - D_4 k^4 + O(k^5) \]
  \[ D_2 = 1 + bc - \frac{2(1 + c^2)q^2}{1 - q^2} \] - phase diffusion

- For $D_2 < 0$ we obtain KS eq (for $q=0$)
  \[ \partial_t \varphi = D_2 \Delta \varphi - D_4 \Delta^2 \varphi + (c - b)(\nabla \varphi)^2, \quad D_2 < 0 \]
Kuramoto-Sivashinsky Equation

• After normalization

\[ \frac{\partial_t \varphi}{\varphi} = -\Delta \varphi - \Delta^2 \varphi + (\nabla \varphi)^2 \]

• Finite wave length linear instability

\[ \varphi \sim \exp[\lambda t + ikx] \]

\[ \lambda = k^2 - k^4 \]

• Periodic solutions are unstable in a large domain
• KSE possesses chaotic solutions
• Describes phase chaos of the CGLE
Very Rich Behavior even in 1D

CI-core instability of NB holes
BFN-Benjamin-Feir line
DC- defect chaos
EH,AH- Eckhaus and absolute instability of NB holes

bi-chaos

phase turbulence  amplitude (defect) turbulence
Conclusions

• The CGLE describes a broad range of phenomena on qualitative and quantitative level

• The CGLE is universal model which can be rigorously derived from various physical, chemical or biological system

• The CGLE is a computationally efficient model and admits analytical treatment
Open Problems

• Understanding of the statistical properties of spatio-temporal chaos in all dimensions
• Elaboration of the vortex glass to vortex liquid transition, properties of glassy states
• Exploration of 3D behavior, from turbulence to helices
• Subcritical CGLE