Stochastic Equations and Processes in Physics and Biology

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Berlin, June 2012

Course outline

- **Introductory lecture**
- **Basic concepts and definitions: Power spectrum and correlations, different types of stochastic processes, examples**
- **Solving stochastic ODEs: the Wiener process, the Ito and Stratonovich interpretation, numerical integration of ODEs, the Fokker-Planck equation.**
- **Brownian Ratchets: general theory, examples involving asymptotic expansion**
- **The Fokker-Planck equation: eignefunction expansion and the linear response theory**
- **Renewal processes and continuous time random walk**
- **Collective phenomena in stochastic networks**

Recommended literature

[1] **Crispin Gardiner** *Stochastic Methods: A Handbook for the Natural and Social Sciences*

[2] **Hannes Risken** *The Fokker-Planck Equation*

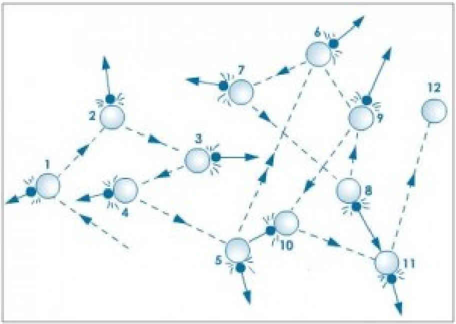
[3] **R.L. Stratonovich** *Topics in the Theory of Random Noise*

[4] **R. Kubo, M. Toda, N. Hashitsume,**
Statistical Physics II

Google on *Stochastic Differential Equations Lecture Notes* **gives over 1.000.000 results**

When fluctuations become important: historical overview

(I) Brownian motion: experiments by Robert Brown (1827) using small polen grains suspended in water



- *Über die von der molekular-kinetischen Theorie der Wärme geforderte Bewegung von in der ruhenden Flüssigkeiten suspendierten Teilchen,* **Albert Einstein Ann. Phys. (Leipzig) 17, 549 (1905)**

- *Zur kinetischen Theorie der Brownsche Bewegung* **Marian Smoluchowski, Ann. Phys. (Leipzig) 21, 756 (1906)**

When fluctuations become important: historical overview

Einstein's reasonings: *(from 1905 paper, translated by C. Risken)*
Let there be a total of n particles (polen grains) suspended in liquid. In a time interval τ, the X-coordinates of the individual particles will increase by an amount Δ, where for each particle Δ has a different value. There will be a certain frequency law for Δ; the number dn of the particles which experience a shift which is between Δ and $\Delta + d\Delta$ will be expressible by an equation of the form

$$dn = n\phi(\Delta)d\Delta,$$

where

$$\int_{-\infty}^{\infty} \phi(\Delta)d\Delta = 1$$

and ϕ is only different from zero for very small values of Δ and satisfies

$$\phi(\Delta) = \phi(-\Delta).$$

We now investigate how the diffusion coefficient depends on ϕ. Let $\nu = f(x,t)$ be the number of particles per unit volume. We compute the distribution of particles at the time $t+\tau$ from the distribution at time t. One obtains

$f(x,t+\tau) = dx \int_{-\infty}^{\infty} f(x+\Delta,t)\phi(\Delta)d\Delta$ **(the Chapman-Kolmogorov equation)**

But since τ is very small, we can set

$$f(x,t+\tau) = f(x,t) + \tau\frac{\partial f}{\partial t}.$$

Furthermore, we develop $f(x+\Delta,t)$ in powers of Δ:

$$f(x+\Delta,t) = f(x,t) + \Delta\frac{\partial f(x,t)}{\partial x} + \frac{\Delta^2}{2}\frac{\partial^2 f(x,t)}{\partial x^2}\dots$$

We can use this series under the integral, because only small values of Δ contribute to this equation. We obtain

$$f + \tau\frac{\partial f}{\partial t} = f\int_{-\infty}^{\infty}\phi(\Delta)d\Delta + \frac{\partial f}{\partial x}\int_{-\infty}^{\infty}\Delta\phi(\Delta)d\Delta + \frac{\partial^2 f}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta$$

Because $\phi(\Delta) = \phi(-\Delta)$, the the second, fourth, etc. terms on the right-hand-side vanish, while out of the 1st, 3rd, 5th, etc., terms, each one is very small compared with the previous. We obtain from this equation, by taking into consideration

$$\int_{-\infty}^{\infty} \phi(\Delta) d\Delta = 1,$$

and setting

$$\frac{1}{\tau} \int_{-\infty}^{\infty} \frac{\Delta^2}{2} \phi(\Delta) d\Delta = D,$$

and keeping only the 1st and third terms

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2} \ldots$$ **(the Fokker-Planck equation)**

This is already known as the differential equation of diffusion and it can be seen that D is the diffusion coefficient.

When fluctuations become important: historical overview

Birth-Death Processes. Kinetic theory of chemical reactions

Macroscopic equilibrium theory of chemical reactions was developed in 1864 by Cato Maximilian Guldberg and Peter Waage (norwegian mathematicians and chemists). Later, William Lewis in 1916-1918 constructed the kinetic theory of chemical reactions, which is based on the collision theory.

When fluctuations become important: historical overview

The average number of molecules per unit time, undergoing the reaction of association

$$\alpha A + \beta B \to C$$

is proportional to the chemical affinity $[A]^\alpha [B]^\beta$ and to the probability of two colliding particles to overcome a certain activation energy barrier E_a. The reaction rate is then given by

$$r_a = s[A]^\alpha [B]^\beta \exp\left(-\frac{E_a}{kT}\right),$$

where s stands for steric factor (correction factor w.r.t experimental values). Similarly, the dissociation of C into A and B occurs with the rate

$$r_d = s[C] \exp\left(-\frac{E_d}{kT}\right),$$

where E_d is the energy barrier for dissociation.

In equilibrium, $r_a = r_d$, which yields the Law of Mass Action

$$K = \exp\left(-\frac{(E_d - E_a)}{kT}\right) = \frac{[C]}{[A]^\alpha [B]^\beta}$$

Kramers theory of chemical reactions: (H. A. Kramers, 1940)

Two reacting chemicals: X_1 and X_2

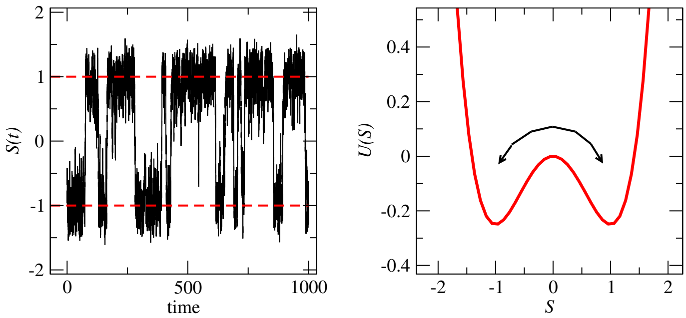
$$X_1 \rightleftharpoons X_2$$

Associated bistable system, described by a stochastic process $S(t) = \pm 1$ with transition probabilities (reaction rates):

$$\lambda_+ = \lim_{\Delta t \to 0} P(S = +1, t + \Delta t | S = -1, t)$$

$$\lambda_- = \lim_{\Delta t \to 0} P(S = -1, t + \Delta t | S = +1, t)$$

$$\dot{S}(t) = -\frac{dU(S)}{dS} + \text{noise}$$



Master equation for number densities C_1 and C_2

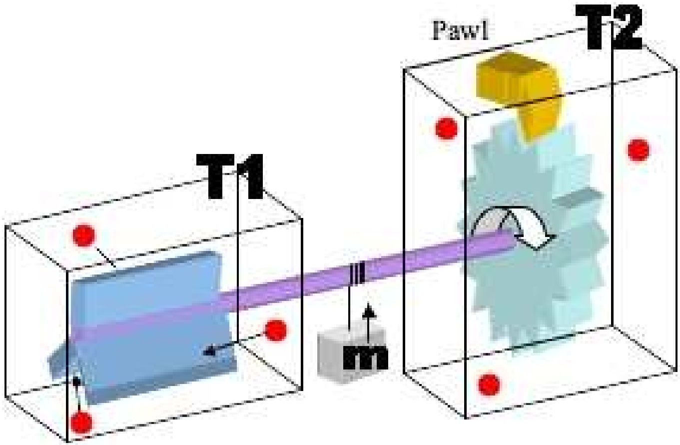
$$\dot{C}_1(t) = -\lambda_+ C_1 + \lambda_- C_2$$
$$\dot{C}_2(t) = -\lambda_- C_2 + \lambda_+ C_1$$

Kramer's formula for the reactions rates

$$\lambda_\pm = \frac{1}{2\pi}\sqrt{\mp\partial_x^2 U(-1)\partial_x^2 U(0)}\exp\left(-\frac{\Delta U}{4D}\right)$$

When fluctuations become important: historical overview

Smoluchowski-Feynamn ratchet (Richard Feynman, 1962)

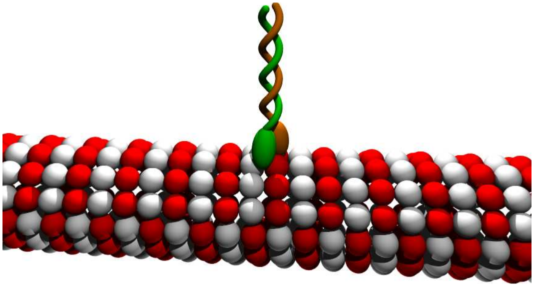


No rotation if in equilibrium $T_1 = T_2$.

Biological examples of Rectified Brownian motion

Kinesin a protein belonging to a class of motor proteins found in eukaryotic (containing a nucleus) cells. Kinesins move along microtubule filaments, and are powered by the hydrolysis of ATP

Kinesin dimer attached to a microtubule

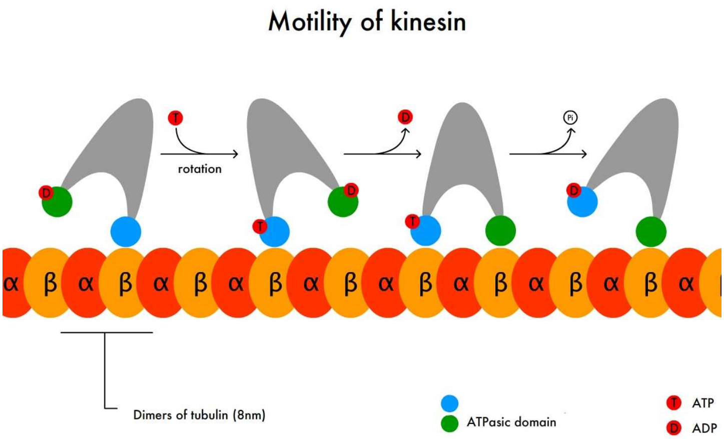


weight: $>$ 100 KD (1 Da $=$ 1.6 $\times 10^{-27}$ kg), size: up to 100 nm

Forward motion of kinesin as Rectified Brownian Motion

Kinesin's Biased Stepping Mechanism: Amplification of Neck Linker Zippering, W. H. Mather and R. F. Fox, Biophys J. (2006) 91(7): 2416–2426.

Motility of kinesin powered by ATP



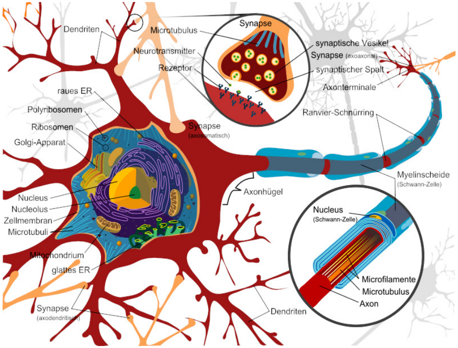
Essential building blocks of the forward motion of kinesin

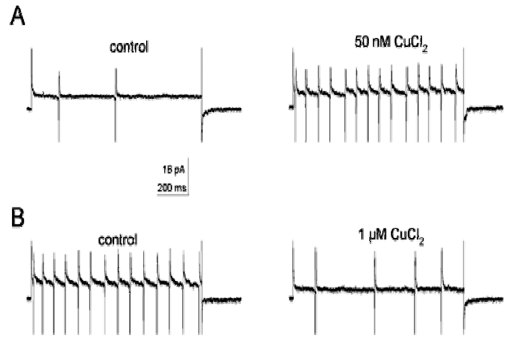
- **Two sources of energy: (1) Neck linker zippering $e \sim 2kT$ and (2) binding of ATP $e \gg kT$, Pulling force $\sim 1.0 \ldots 7.0$ pN**
- **Directed cargo transport is the result of the diffusional displacement of the heads, biased by small-energy zippering and fueled by large-energy ATP binding.**

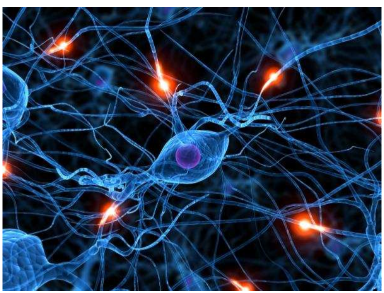
Conditions for rectified Brownian Motion

- **Broken spatial symmetry**
- **Fluctuations (noise)**
- **Out of equilibrium due to external energy supply**
- **Flashing ratchet (on and off ratchets)**

Spiking Neurons and Neural Networks

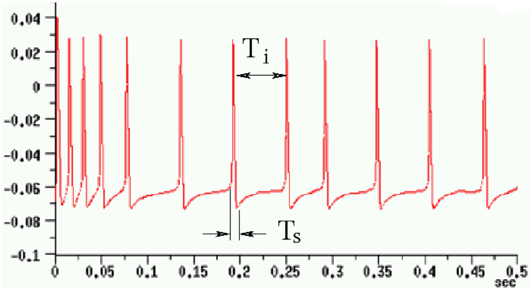






- Each neuron receives signals from other neurons through dendrites
- An electrical pulse is fired along the axon if the integral input signal exceeds a threshold

Neuron as an excitable system



- Spike durations T_s are fixed
- Inter-spike intervals T_i are random

Models of the electrical activity of a neuron

- **Hodgkin-Huxley model is an electric circuit model of a neuron** (Nobel Price in Physiology or Medicine (1963))
- **The FitzHugh-Nagumo model** (2D dynamical system (1961))
- **Integrate-and-fire models** (1D models)

Origin of fluctuations

- **Molecular motion: thermal fluctuations**
- **Neurons: random synaptic input from other neurons, quasi-random release of neurotrasmitter by the synapses, random switching of ion channels**
- **Chemical reactions: finite-size effects**
- **Lasers: quantum fluctuations, the uncertainty principle**
- **Weather: complexity, chaos**

On the classical level: Randomness=Chaos