I. RECTIFIED BROWNIAN MOTION: BROWNIAN RATCHETS

The main idea is to create directed motion facilitated by thermal fluctuations.

Theoretical framework

Consider a (asymmetric) one-dimensional periodic potential $U(x)$ with period L, i.e. $U(x+L) = U(x)$. Typically, one uses the following biharmonic function $U(x) = \sin 2\pi x + (1/4)\sin 4\pi x$. The motion of a point particle m in $U(x)$ is then given by

$$\begin{aligned} \dot{x} &= v \\ m\dot{v} &= -\gamma v - U'(x) + F(t) + \sqrt{2\gamma m k T}\xi(t), \end{aligned} \tag{1}$$

where $F(t)$ is some external generally time-dependent force and T stands for the temperature of thermal bath. In what follows, we assume that $F(t)$ is periodic with period T and unbiased, i.e. $\int_t^{t+T} F(t)\,dt = 0$.

In the overdamped limit, i.e. assuming that $m/\gamma \to 0$, it is possible to show that the dynamics of the particle is described by a single first-order equation

$$\dot{x} = -U'(x) + F(t) + \sqrt{2D}\xi(t). \tag{2}$$

The corresponding Fokker-Planck equation reads

$$\partial_t \rho = \partial_x[(U' - F)\rho + D\partial_x \rho] = -\partial j(x,t), \tag{3}$$

with the probability current density $j(x,t) = -(U' - F)\rho - D\partial_x \rho$.

Rocked ratchets: the adiabatic current

We are interested in the average current $\bar{J}$, given by

$$\bar{J} = \lim_{t\to\infty} \frac{1}{t} \int_x^{x+L} j(x',t)\,dx'\,dt. \tag{4}$$

Analytical results exist for the adiabatic limit, when $\tau \to \infty$.

In order to compute the average current in the adiabatic limit, we first determine the stationary current for a constant drive $F = const$.

The solution of the stationary F-P equation can be written as

$$\rho_s(x) = Ce^{(-U_{\rm eff}(x)/D)} - \frac{J}{D} e^{(-U_{\rm eff}(x)/D)} \int_0^x e^{(U_{\rm eff}(y))/D}\,dy, \tag{5}$$

with the effective potential $U_{\rm eff}(x) = U(x) - Fx$ ad two unknown constants C and J. Note that the constant J is in fact the stationary current.

The unknown constants can be found from the normalization condition on the density and from the fact that $\rho_s(x)$ is periodic with the period given by L.

The periodicity of $\rho_s(x)$ requires

$$\begin{aligned} & Ce^{(-U_{\text{eff}}(x+L)/D)} - \frac{J}{D} e^{(-U_{\text{eff}}(x+L)/D)} \int_0^{x+L} e^{(U_{\text{eff}}(y))/D}\, dy \\ = \; & Ce^{(-U_{\text{eff}}(x)/D)} - \frac{J}{D} e^{(-U_{\text{eff}}(x)/D)} \int_0^{x} e^{(U_{\text{eff}}(y))/D}\, dy \end{aligned} \tag{6}$$

Equivalently

$$C[e^{(FL/D)} - 1] - \frac{J}{D} e^{(FL/D)} \left[I(x+L) - I(0) \right] = -\frac{J}{D} [I(x) - I(0)], \tag{7}$$

where $I(x)$ denotes the indefinite integral $I(x) = \int e^{(U_{\text{eff}}(y))/D}\, dy$.

By differentiation, one can show that the function

$$g(x) = e^{(FL/D)} I(x+L) - I(x) \tag{8}$$

is in fact a constant, i.e. $g' = 0$. Therefore, without loss of generality, we can choose $x = 0$ in Eq. (7) and obtain

$$C = \frac{J e^{(FL/D)} [I(L) - I(0)]}{D[e^{(FL/D)} - 1]} \tag{9}$$

Finally, the stationary current J is found from the normalization condition

$$\int_x^{x+L} \rho(x')\, dx' = 1. \tag{10}$$

This condition can be written in terms of the two new functions

$$\begin{aligned} I^-(x) &= \int_0^x \exp\left(-U_{\text{eff}}(y)/D\right) dy \\ I^+(x) &= \int_0^x \exp\left(U_{\text{eff}}(y)/D\right) dy = I(x) - I(0) \end{aligned} \tag{11}$$

By setting $x = 0$ in Eq. (10), we obtain

$$C I^-(L) - \frac{J}{D} \int_0^L e^{(-U_{\text{eff}}(x)/D)} [I(x) - I(0)]\, dx = 1. \tag{12}$$

Consequently,

$$J = \frac{D(e^{(FL/D)} - 1)}{I^+(L) I^-(L) e^{(FL/D)} - (e^{(FL/D)} - 1) \int_0^L e^{(-U_{\text{eff}}(x)/D)} I^+(x)\, dx} \tag{13}$$

Typical dependence of J on F and D is shown in Fig. 1. Any asymmetric ratchet potential

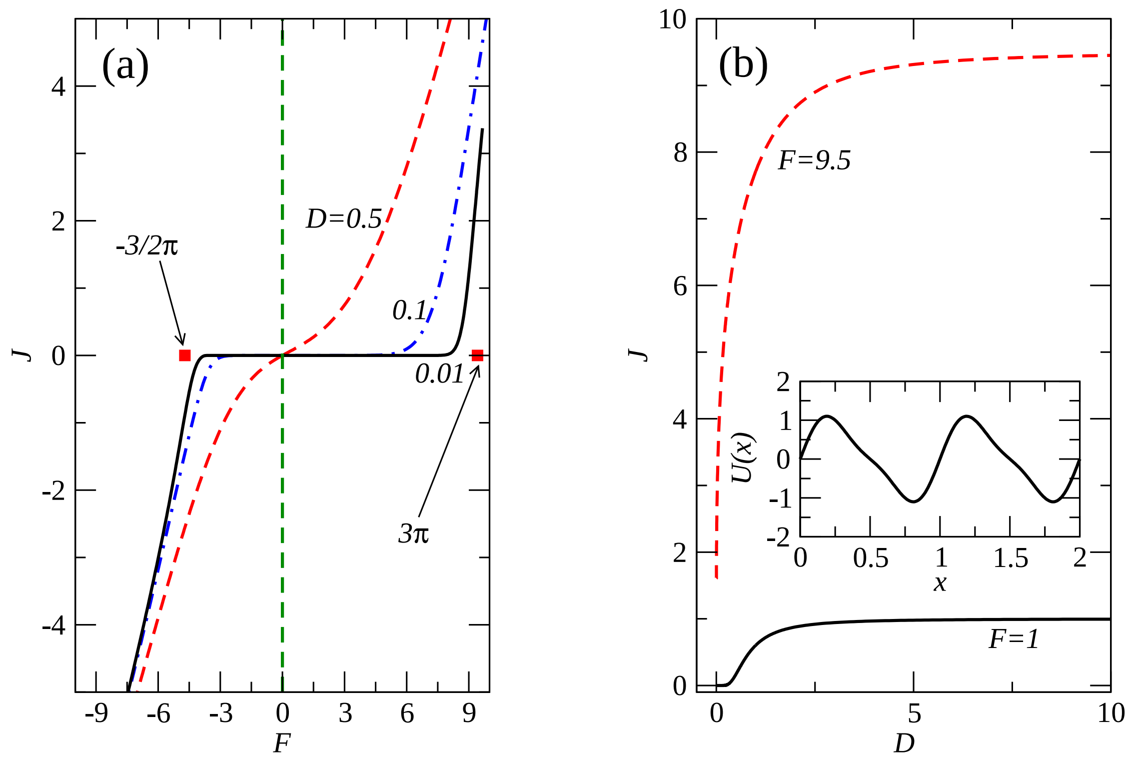

FIG. 1: (a) Stationary current J *vs* F for different values of D as in the legend. The left and right depinning forces $F_L = -3\pi/2$ and $F_R = 3\pi$ are shown by symbols on the horizontal axis. (b) J as a function of D for two different F as in the legend. Inset of (b): the ratchet potential $U(x) = \sin 2\pi x + 0.25 \sin 4\pi x$.

$U(x)$ is characterized by the left- and right depinning forces F_L and F_R, respectively. These determine the critical values of the external constant force, which is induces the unbounded motion of the particle in the negative and positive directions, respectively. For the standard biharmonic potential $U(x) = \sin 2\pi x + 0.25 \sin 4\pi x$, the depinning forces are given by $F_L = -3\pi/2$ and $F_R = 3\pi$, implying that the positive direction is the so-called "hard" direction (see the inset of Fig. 1(b)).

Consequently, in the limit of $D \to 0$, the current remains practically zero for $F \in [F_L, F_R]$, as shown in Fig. 1(a). If the noise intensity $D \neq 0$, the current is non-zero at any F. Important is that due to the asymmetry of the ratchet potential, the function $J(F)$ is *not an odd function*, implying that $J(F) \neq -J(-F)$. It is also worthwhile noticing that at any fixed F the current J approaches F as $D \to \infty$, as demonstrated in Fig. 1(b).

We are now ready to compute the adiabatic average current $\langle J \rangle$ for the unbiased time-

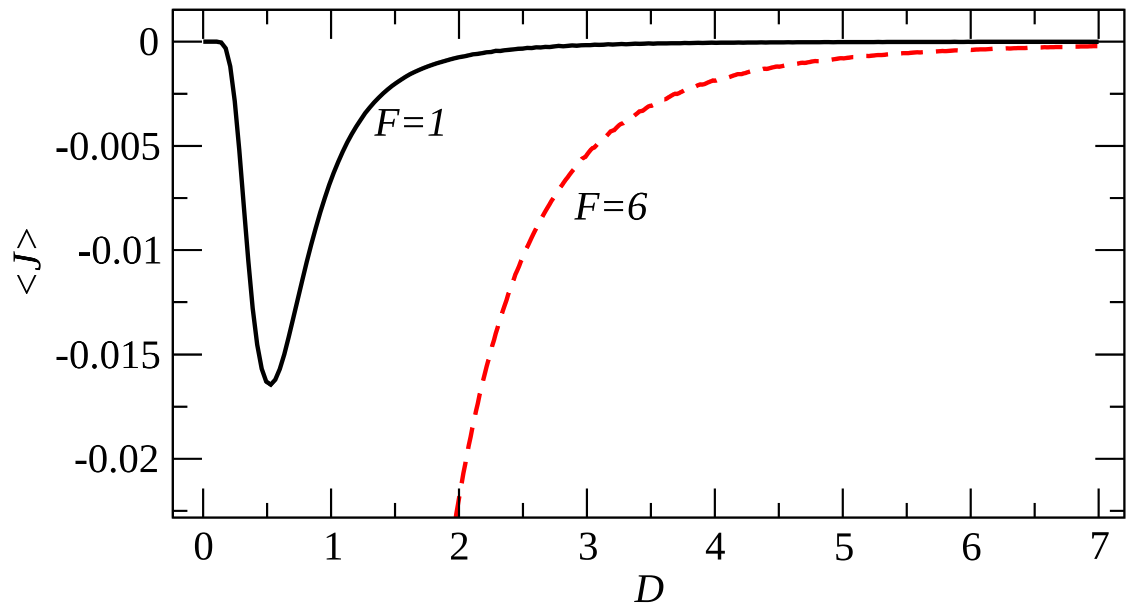

FIG. 2: Average adiabatic current $\langle J\rangle$ *vs* D for different values of the amplitude F as in the legend.

periodic drive $F(t)$ with vanishing frequency (infinitely large period). As such, we take the square-wave drive, which is given by $F(t) = \pm F$, where each value is kept over the half of the total period T.

Clearly, in this case, the average current is given by

$$\langle J\rangle = \frac{J(F)T/2 + J(-F)T/2}{T} = \frac{J(F) + J(-F)}{2}. \tag{14}$$

Typical dependence of the average adiabatic current $\langle J\rangle$ as a function of D is shown in Fig. 2. In the sub-critical regime, i.e. when $\mid F \mid<\mid F_L \mid$, the current is noise-induced and it disappears as $D \to \infty$.

Translocation ratchet: translocation of polymers

following C. S. Peskin et al, "Cellular Motion and Thermal Fluctuations: The Brownian Ratchet", Biophysical Journal **65** *316-324 (1993)*

Consider a protein, passing through a translocation pore. It is modeled by a long rod, which diffuses along the x axis. The rod is made of identical sections (segments) of the length δ. The pore acts as a perfect ratchet, only allowing the motion to the right. The flux of the rod, driven by the force f is given by

$$\phi = -\mu f\rho - D\partial_x\rho, \tag{15}$$

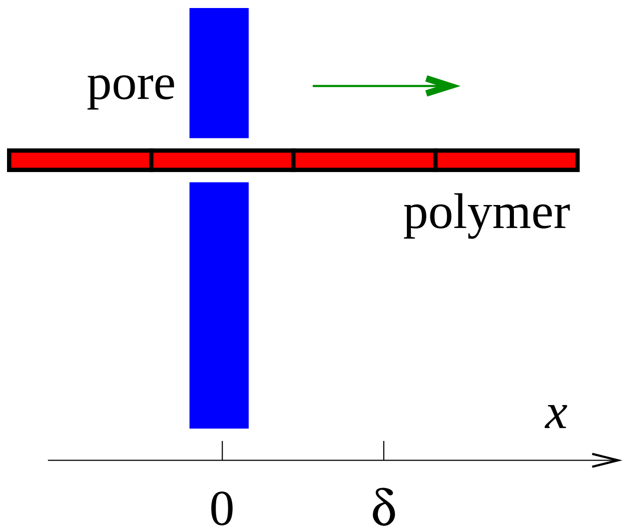

FIG. 3: Schematic diagram of the translocation ratchet. Shown is the pore with a segmented polymer inside, moving to the right. The length of each segment is δ.

where ρ is the density, μ is the mobility of the rod and D is the diffusion coefficient. Note that the force f acts against the translocation direction.

The Fokker-Planck equation

$$\partial_t \rho + \partial_x \phi = 0 \tag{16}$$

is supplemented by the following boundary conditions

$$\phi(0,t) = \phi(\delta,t), \quad \rho(\delta,t) = 0. \tag{17}$$

These boundary conditions imply that we consider the motion of one segment between $x = 0$ and $x = \delta$. Each time a segment of the rod reaches the point $x = \delta$, it is removed from the system and is replaced by the new section at the point $x = 0$. In the stationary state, the flux ϕ is constant and the density ρ_s can be found exactly

$$\rho_s(x) = Ae^{(-\mu f x/D)} - \frac{\phi}{\mu f}, \tag{18}$$

with some unknown A and the unknown flux ϕ. From the second BC, we get

$$A = \frac{\phi}{\mu f} e^{(\mu f \delta/D)}. \tag{19}$$

The first BC holds automatically, because $\phi = const$.

We are interested in the average translocation velocity v, which is defined as the flux ϕ, normalized to a single segment, multiplied by δ, i.e. $v = \delta\phi$, where the normalization condition is given by

$$\int_0^\delta \rho_s(x)\, dx \;=\; 1, \quad \text{or} \tag{20}$$

$$\frac{\phi}{\mu f}\left[\frac{D}{\mu f}\left(e^{\mu f \delta/D} - 1\right) - \delta\right] \;=\; 1 \tag{21}$$

Finally,

$$v = \frac{D}{\delta}\frac{\omega^2}{(e^\omega - 1) - \omega}, \tag{22}$$

with $\omega = \delta\mu f/D$. Interestingly, for vanishing forcing $f = 0$, the translocation velocity is $V = 2D/\delta$. The velocity v decreases monotonically with ω and reaches zero $v = 0$ at a certain stall load f_c.

For imperfect translocation ratchet as well as the polymerization ratchets, see the original paper.

Thermal ratchets, the ratchet effect

Consider a so-called flashing or thermal ratchet, where the ratchet potential changes periodically in time (on and off ratchet). An overdamped Brownian particle will drift on average in the "hard" direction, even if a small external force is applied against this motion. This phenomenon is called the ratchet effect.

The fundamental characteristic of a thermal ratchet is the dependence of the average current $\langle J\rangle$ on the flashing frequency ω. It has been shown that in case when the external force is zero, the average particle current vanishes in both limits, when $\omega \to 0$ and $\omega \to \infty$ and reaches a maximum at a certain critical frequency ω_c.

Our aim is now to derive the asymptotic decay law for $\langle J\rangle$ in the limit of fast oscillations, i.e. when $\omega \to \infty$. To this end, we are following the original paper by

P. Reimann et al, "Brownian motors driven by temperature oscillations", Physics Letters A **215**, *26-31 (1996).*

Consider an overdamped particle in the ratchet potential $V(x)$ with time-modulated temperature of the thermal bath

$$T(t) = \bar{T} + \Delta(t), \tag{23}$$

with unbiased periodic modulation $\Delta(t + 2\pi/\omega) = \Delta(t)$ and $\int_t^{t+2\pi/\omega} \Delta(t)\, dt = 0$.

The Fokker-Planck equation is then given by

$$\partial_t \rho = \partial_x [V'(x)\rho + (\bar{T} + \Delta(t))\partial_x \rho]. \tag{24}$$

In order to study the limit $\omega \to \infty$, we rescale time according to $t' = t\omega$. The transformed F-P equation is

$$\partial_{t'} \rho = \frac{1}{\omega}\partial_x \left[V'(x)\rho + \bar{T}\partial_x \rho + \Delta\left(\frac{t'}{\omega}\right)\partial_x \rho \right]. \tag{25}$$

The rescaled modulation is normalized according to

$$\int_0^{2\pi} \Delta'(t')\, dt' = 0. \tag{26}$$

In what follows we set for simplicity $t' \to t$.

Following standard methods of asymptotic analysis, we represent the time-dependent density ρ as an infinite series of the form

$$\rho(x,t) = \rho_0(x,t) + \frac{1}{\omega}\rho_1(x,t) + \frac{1}{\omega^2}\rho_2(x,t) + \ldots, \tag{27}$$

where each function $\rho_i(x,t)$ is of the order of $O(\omega^0)$.

It is clear that this expansion implies the following normalization for ρ_i

$$\int_x^{x+L} \rho_i(x,t)\, dx = \delta_{i,0}. \tag{28}$$

Substituting the expansion Eq. (27) into the rescaled F-P equation Eq. (25), and comparing the terms of the same order of $(1/\omega)^i$, we obtain a sequence of coupled equations for $\rho_i(x,t)$. Thus, in the zeroth order, we get

$$\left(\frac{1}{\omega}\right)^0 : \quad \frac{\partial \rho_0}{\partial t} = 0, \tag{29}$$

implying that $\rho_0 = \rho_0(x)$. The next order gives

$$\left(\frac{1}{\omega}\right)^1 : \quad \frac{\partial \rho_1}{\partial t} = \frac{\partial}{\partial x}\left(V'\rho_0 + \bar{T}\frac{\partial \rho_0}{\partial x} + \Delta(t)\frac{\partial \rho_0}{\partial x} \right). \tag{30}$$

By integrating Eq. (30) over one period of the forcing and taking into account that

$$\int_0^{2\pi} \frac{\partial \rho_1(x,t)}{\partial t}\, dt = 0, \tag{31}$$

we obtain

$$0 = \frac{\partial}{\partial x}\left(V'\rho_0 + \bar{T}\frac{\partial \rho_0}{\partial x} \right), \tag{32}$$

and, consequently

$$\rho_0(x) = e^{\left(-\frac{V(x)}{\bar{T}}\right)} / \left[\int_0^L e^{\left(-\frac{V(x)}{\bar{T}}\right)}\, dx\right]. \tag{33}$$

From Eq. (30) we can now find ρ_1

$$\rho_1(x,t) = I(t)\rho_o'' + f_1(x), \tag{34}$$

where $I(t)$ is given by the indefinite integral

$$I(t) = \int \Delta(t)\, dt \tag{35}$$

and the function $f_1(x)$ is time-independent.

It is important to notice that by integrating the F-P equation Eq. (25) over one period of the forcing, and using the fact that $\int_0^{2\pi} \partial_t \rho(x,t)\, dt = 0$, we conclude that the time-averaged current density is x-independent, i.e.

$$\bar{J} = \frac{1}{2\pi}\int_0^{2\pi} j(x,t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} \left[-V'\rho - \bar{T}\partial_x\rho - \Delta(t)\partial_x\rho\right]\, dt = const. \tag{36}$$

This should hold in any order of $(1/\omega)^i$.

The current density in the order of $(1/\omega)^1$ is given by

$$j_1(x,t) = -V'\rho_1 - \bar{T}\partial_x\rho_1 - \Delta(t)\partial_x\rho_1. \tag{37}$$

In order to find the time-averaged density, we use the following result

$$\int \Delta(t) I(t)\, dt = \frac{I^2(t)}{2}. \tag{38}$$

Noticing that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \rho_1\, dt &= f_1(x), \\ \frac{1}{2\pi}\int_0^{2\pi} \Delta(t)\rho_1\, dt &= \rho_0''\frac{1}{2\pi}\int_0^{2\pi} \Delta(t)I(t)\, dt = \frac{\rho_0''}{2\pi}\left(\frac{I^2(2\pi)}{2} - \frac{I^2(0)}{2}\right) = 0, \end{aligned} \tag{39}$$

we obtain the time-averaged current in the order of $(1/\omega)^1$

$$\bar{J}_1 = -V' f_1(x) - \bar{T}\frac{\partial f_1(x)}{\partial x} = \mu_1 = const. \tag{40}$$

The solution of the last equation is

$$f_1(x) = Ce^{(-V(x)/\bar{T})} - \frac{\mu_1}{\bar{T}} e^{(-V(x)/\bar{T})}\int_0^x e^{(V(y)/\bar{T})}\, dy. \tag{41}$$

We need to impose two additional conditions on f_1, namely, the periodicity and the normalization conditions

$$f_1(x+L) = f_1(x), \quad \int_0^L f_1(x)\,dx = 0, \tag{42}$$

The periodicity implies

$$\frac{\mu_1}{\bar{T}} \int_x^{x+L} e^{(V(y)/\bar{T})}\,dy = 0, \tag{43}$$

which is only true if $\mu_1 = 0$. The normalization yields $C = 0$, and, consequently $f_1(x) = 0$.

As we see, the flux in the order of $(1/\omega)^1$, given by μ_1 is zero. Therefore, we proceed to the next order.

$$\left(\frac{1}{\omega}\right)^2 : \quad \frac{\partial \rho_2}{\partial t} = \frac{\partial}{\partial x}\left[V'\rho_1 + \bar{T}\frac{\partial \rho_1}{\partial x} + \Delta(t)\frac{\partial \rho_1}{\partial x}\right]. \tag{44}$$

Or, equivalently

$$\frac{\partial \rho_2}{\partial t} = \frac{\partial}{\partial x}\left[V'\rho_0''I(t) + \bar{T}I(t)\rho_0''' + \Delta(t)I(t)\rho_0'''\right]. \tag{45}$$

This gives the expression for $\rho_2(x,t)$

$$\rho_2(x,t) = \frac{\partial}{\partial x}[V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}]I_2(t) + \frac{I^2(t)}{2}\rho_0^{(4)} + f_2(x), \tag{46}$$

where $f_2(x)$ depends only on x and the function $I_2(t)$ is given by the indefinite integral

$$I_2(t) = \int I(t)\,dt. \tag{47}$$

The time-averaged current in the order of $(1/\omega)^2$ is

$$\langle J\rangle_2 = -V'\langle\rho_2\rangle - \bar{T}\partial_x\langle\rho_2\rangle - \langle\Delta(t)\partial_x\rho_2\rangle, \tag{48}$$

where $\langle\ldots\rangle$ denotes the time-average.

In order to explicitly compute the time-average values, we make use of the following auxiliary results

$$\begin{aligned}
\langle\rho_2\rangle &= \rho_0^{(4)}\frac{\langle I^2(t)\rangle}{2} + f_2(x),\\
\langle\Delta(t)\rho_2\rangle &= \frac{\partial}{\partial x}\left(V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right)\frac{1}{2\pi}\int_0^{2\pi}\Delta(t)I_2(t)\,dt + \rho_0^{(4)}\frac{1}{2\pi}\int_0^{2\pi}\Delta(t)\frac{I^2(t)}{2}\,dt\\
&= -\frac{\partial}{\partial x}\left(V'\rho_o^{(2)} + \bar{T}\rho_0^{(3)}\right)\langle I^2(t)\rangle,
\end{aligned} \tag{49}$$

where we have used the fact that $\langle I(t)\rangle = (1/2\pi)[I_2(2\pi) - I_2(0)] = 0$ and that

$$\begin{aligned}
\int \Delta(t) I_2(t)\, dt &= I(t)I_2(t) - \int I^2(t)\, dt, \\
\frac{1}{2\pi}\int_0^{2\pi} \Delta(t) I_2(t)\, dt &= -\frac{1}{2\pi}\int_0^{2\pi} I^2(t)\, dt, \\
\int_0^{2\pi} \Delta(t) \frac{I^2(t)}{2}\, dt &= \frac{I^3(t)}{6}\Big|_0^{2\pi} = 0.
\end{aligned} \tag{50}$$

Therefore, the time-averaged current becomes

$$\begin{aligned}
\bar{J}_2 &= -\left(V' f_2 + \bar{T}\frac{\partial f_2}{\partial x}\right) - \frac{\langle I^2(t)\rangle}{2}\left(V'\rho_0^{(4)} + \bar{T}\rho_0^{(5)}\right) \\
&+ \langle I^2(t)\rangle \frac{\partial^2}{\partial x^2}\left(V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right) = \mu_2 = const.
\end{aligned} \tag{51}$$

The last equation can be written in a slightly more compact form

$$\left(V' f_2 + \bar{T}\frac{\partial f_2}{\partial x}\right) = -\mu_2 + g(x), \tag{52}$$

with

$$g(x) = \frac{\langle I^2(t)\rangle}{2}\left(V'\rho_0^{(4)} + \bar{T}\rho_0^{(5)}\right) - \langle I^2(t)\rangle \frac{\partial^2}{\partial x^2}\left(V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right). \tag{53}$$

The solution for $f_2(x)$ is

$$f_2(x) = Ce^{\left(-\frac{V(x)}{\bar{T}}\right)} + \frac{1}{\bar{T}} e^{\left(-\frac{V(x)}{\bar{T}}\right)} \int_0^x [g(y) - \mu_2] e^{\left(\frac{V(y)}{\bar{T}}\right)}\, dy. \tag{54}$$

From the periodicity of $f_2(x)$ we get

$$\mu_2 = \frac{1}{I^+}\int_x^{x+L} g(y) e^{\left(\frac{V(y)}{\bar{T}}\right)}\, dy, \tag{55}$$

with $I^+ = \int_x^{x+L} e^{\left(\frac{V(y)}{\bar{T}}\right)}\, dy$.

Explicitly, we have

$$\mu_2 = \frac{1}{I^+}\int_x^{x+L}\left[\frac{\langle I^2(t)\rangle}{2}\left(V'\rho_0^{(4)} + \bar{T}\rho_0^{(5)}\right) - \langle I^2(t)\rangle \frac{\partial^2}{\partial x^2}\left(V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right)\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx. \tag{56}$$

The first term in Eq. (56) in the square brackets can be computed using integration by parts

$$\int_x^{x+L}\left[V'\rho_0^{(4)} + \bar{T}\rho_0^{(5)}\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx = \int_x^{x+L}\left[V'\rho_0^{(4)} - \frac{1}{\bar{T}}\bar{T}\rho_0^{(4)}\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx = 0. \tag{57}$$

The second term in Eq. (56) in the square brackets is transformed by integrating it two times by parts

$$\int_x^{x+L}\frac{\partial^2}{\partial x^2}\left[V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx = \int_x^{x+L}\left(\frac{V''}{\bar{T}} + \frac{V'^2}{\bar{T}^2}\right)\left[V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx. \tag{58}$$

Now recalling that

$$\begin{aligned}
\rho_0' &= -\left(\frac{1}{I^-}\right)\frac{V'}{\bar{T}}e^{\left(-\frac{V}{\bar{T}}\right)}, \\
\rho_0^{(2)} &= \left(\frac{1}{I^-}\right)\left(-\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)e^{\left(-\frac{V}{\bar{T}}\right)}, \\
\rho_0^{(3)} &= \left(\frac{1}{I^-}\right)\left(-\frac{V^{(3)}}{\bar{T}}+3\frac{V^{(2)}V^{(1)}}{\bar{T}^2}-\frac{V'^3}{\bar{T}^3}\right)e^{\left(-\frac{V}{\bar{T}}\right)},
\end{aligned} \tag{59}$$

with $I^- = \int_0^L e^{\left(-\frac{V(y)}{\bar{T}}\right)}\,dy$, we obtain

$$\int_x^{x+L}\left(\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)\left[V'\rho_0^{(2)}+\bar{T}\rho_0^{(3)}\right]e^{\left(\frac{V}{\bar{T}}\right)}\,dx \tag{60}$$

$$\begin{aligned}
&= \frac{1}{I^-}\int_x^{x+L}\left(\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)\left[V'\left(-\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)+\bar{T}\left(-\frac{V^{(3)}}{\bar{T}}+3\frac{V^{(2)}V^{(1)}}{\bar{T}^2}-\frac{V'^3}{\bar{T}^3}\right)\right]dx \\
&= \frac{1}{I^-}\int_x^{x+L}\left(\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)\left[-V^{(3)}+2\frac{V^{(2)}V^{(1)}}{\bar{T}}\right]dx.
\end{aligned} \tag{61}$$

The last integral can be simplified using the following results

$$\begin{aligned}
\int_x^{x+L} V''V^{(3)}\,dx &= \frac{V''^2}{2}\Big|_x^{x+L} = 0, \\
\int_x^{x+L} V'^3V^{(2)}\,dx &= \frac{V'^4}{4}\Big|_x^{x+L} = 0, \\
\int_x^{x+L} V'^2V^{(3)}\,dx &= -2\int_x^{x+L} V'V''^2\,dx.
\end{aligned} \tag{62}$$

This yields

$$\int_x^{x+L}\left(\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)\left[V'\rho_0^{(2)}+\bar{T}\rho_0^{(3)}\right]e^{\left(\frac{V}{\bar{T}}\right)}\,dx = \frac{4}{\bar{T}^2I^-}\int_x^{x+L} dx\,V'V''^2. \tag{63}$$

Finally, we obtain the main result of our calculations, i.e. the average current in the second order of $(1/\omega)$

$$\mu_2 = -\frac{1}{\omega^2}\frac{2\int_0^{2\pi} I^2(t)\,dt}{\pi\bar{T}^2I^-I^+}\int_x^{x+L} dx\,V'V''^2. \tag{64}$$