I. EIGENFUNCTIONS OF THE FOKKER-PLANCK OPERATOR

Transformation to a Schrödinger equation

Consider the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left[V'\rho + T\frac{\partial \rho}{\partial x}\right], \tag{1}$$

with arbitrary $V(x)$. The r.h.s. of the last equation can be written as in terms of the Fokker-Planck operator $\hat{L}$

$$\hat{L}\rho = [V'' + V'\partial_x + T\partial_x^2]\rho. \tag{2}$$

Note that the operator $\hat{L}$ is non-Hermitian, because of the first derivative ∂_x.

Recall that the momentum operator $-ih\partial_x$ is Hermitian.

Introduce a new function

$$\phi(x) = \frac{\rho(x)}{\sqrt{\rho_s(x)}}, \tag{3}$$

where $\rho_s(x)$ is the stationary solution of Eq. (1), given by

$$\rho_s(x) = Ce^{\left(-\frac{V(x)}{T}\right)}, \quad C = \left[\int_{-\infty}^{\infty} e^{\left(-\frac{V(x)}{T}\right)}\, dx\right]^{-1}. \tag{4}$$

Then we have

$$\begin{aligned} \rho' &= \left(\phi' - \frac{\phi V'}{2T}\right)\sqrt{C}e^{\left(-\frac{V(x)}{2T}\right)}, \\ \rho'' &= \left(\phi'' - \frac{\phi' V'}{T} - \frac{\phi V''}{2T} + \frac{\phi V'^2}{4T^2}\right)\sqrt{C}e^{\left(-\frac{V(x)}{2T}\right)}. \end{aligned} \tag{5}$$

Therefore, the Eq. (2) becomes

$$\hat{L}\rho = \left[\left(\frac{V''}{2} - \frac{V'^2}{4T}\right)\phi + T\phi''\right]\sqrt{\rho_s}, \tag{6}$$

or,

$$\hat{L}\rho = \sqrt{\rho_s}\hat{H}\phi, \tag{7}$$

where the new Hermitian operator $\hat{H}$ is given by

$$\hat{H} = \left(\frac{V''}{2} - \frac{V'^2}{4T}\right) + T\frac{\partial^2}{\partial x^2}, \tag{8}$$

Now by setting $\rho(x,t) = \phi(x,t)\sqrt{\rho_s(x)}$ in Eq. (1), we obtain the equation for ϕ

$$\frac{\partial \phi}{\partial t} = \hat{H}\phi. \tag{9}$$

Eigenfunction expansion

It is now convenient to represent the density $\rho(x,t)$ as a superposition of the eigenfunctions of $\hat{L}$

$$\rho(x,t|x_0,t_0) = \sum_0^\infty A_n(x_0)\rho_n(x)e^{-\lambda_n(t-t_0)}, \tag{10}$$

where index n numbers the eigenfunctions

$$\hat{L}\rho_n = -\lambda_n \rho_n. \tag{11}$$

Clearly, if $\rho_n = \sqrt{\rho_s}\phi_n$, then ϕ_n solves the eigenvalue problem of the operator $\hat{H}$ with *the same eigenvalues* λ_n

$$\hat{H}\phi_n = -\lambda_n \phi_n. \tag{12}$$

Because $\hat{H}$ is Hermitian, the eigenvalues λ_n are real and the eigenfunctions are orthogonal

$$\int_{-\infty}^{\infty} \phi_n(x)\phi_m(x)\,dx = \delta_{nm}, \tag{13}$$

where δ_{nm} should be replaced by $\delta(n-m)$ for continuous spectrum.

Consequently, the normalization of the functions ρ_n is given by

$$\int_{-\infty}^{\infty} \frac{\rho_n(x)\rho_m(x)}{\rho_s(x)}\,dx = \delta_{nm}, \tag{14}$$

where $\rho_0 = \rho_s$ and $\rho_n = \sqrt{\rho_s}\phi_n$, for $(n = 1, 2, 3, \ldots)$.

Positivity of eigenvalues

Consider the expression

$$\int_{-\infty}^{\infty} \frac{\rho_n(x)\hat{L}\rho_n(x)}{\rho_s(x)}\,dx = \int_{-\infty}^{\infty} \frac{\rho_n(x)(-\lambda_n\rho_n(x))}{\rho_s(x)}\,dx = -\lambda_n. \tag{15}$$

On the other hand, after integrating by parts, we have

$$\int_{-\infty}^{\infty} \frac{\rho_n(x)\hat{L}\rho_n(x)}{\rho_s(x)}\,dx = -\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\frac{\rho_n(x)}{\rho_s}\right)\left\{V'\rho_n + T\frac{\partial \rho_n}{\partial x}\right\}dx \tag{16}$$

But because

$$\frac{\partial}{\partial x}\left(\frac{\rho_n(x)}{\rho_s}\right) = \frac{1}{T\rho_s}\left(T\rho_n' + V'\rho_n\right), \tag{17}$$

we conclude that

$$\int_{-\infty}^{\infty} \frac{\rho_n(x)\hat{L}\rho_n(x)}{\rho_s(x)}\,dx = -\int_{-\infty}^{\infty} T\rho_s\left(\frac{\partial}{\partial x}\frac{\rho_n(x)}{\rho_s}\right)^2 dx = -\lambda_n < 0. \tag{18}$$

Completeness

The completeness condition for the eigenfunctions ϕ_n can be expressed as

$$\sum_n \phi_n(x)\phi_n(x') = \delta(x - x'). \tag{19}$$

This condition can also be written in terms of ρ_n

$$\sum_n \frac{\rho_n(x)\rho_n(x')}{\sqrt{\rho_s(x')}\sqrt{\rho_s(x)}} = \frac{1}{\rho_s(x')}\sum_n \rho_n(x)\rho_n(x') = \delta(x - x'). \tag{20}$$

Transition probability density

Using the completeness condition, we can represent the time-dependent probability $\rho(x, t|x_0, t_0)$ in terms of ρ_n as follows

$$\begin{aligned}\rho(x, t|x_0, t_0) &= e^{\hat{L}(x)(t-t_0)}\delta(x - x_0) = e^{\hat{L}(x)(t-t_0)}\frac{1}{\rho_s(x_0)}\sum_n \rho_n(x)\rho_n(x_0) \\ &= \sum_n e^{-\lambda_n(t-t_0)}\frac{\rho_n(x_0)}{\rho_s(x_0)}\rho_n(x). \end{aligned} \tag{21}$$

Stationary ACF

By definition, the stationary ACF $R(\tau)$ is given by

$$\begin{aligned}R(\tau) &= \int dx\,dx'\,x\,x'\rho(x, t|x_0, t_0)\rho_s(x_0) \\ &= \int dx\,dx'\,x\,x'\sum_n e^{-\lambda_n(t-t_0)}\rho_n(x_0)\rho_n(x) = \sum_n e^{-\lambda_n(t-t_0)}I_n^2, \end{aligned} \tag{22}$$

with

$$I_n = \int_{-\infty}^{\infty} x\rho_n(x)\,dx. \tag{23}$$

Example: Brownian motion with dry friction

[1] de Gennes P-G 2005 "Brownian motion with dry friction" J. Stat. Phys. 119 95362 (2005)

followed by

[2] H. Touchette et al, "Brownian motion with dry friction: FokkerPlanck approach", J. Phys. A **43** *445002 (2010)*

Consider a Brownian particle moving at the presence of a dry-friction force, which is determined by

$$\boldsymbol{F}(\boldsymbol{v}) = \begin{cases} 0, & \boldsymbol{v} = 0 \\ -\gamma \frac{\boldsymbol{v}}{|\boldsymbol{v}|}, & \boldsymbol{v} \neq 0 \end{cases} \tag{24}$$

The 1D motion is then described by the Langevin equation

$$m\dot{v} = -\gamma \mathrm{sgn}(v) + \sqrt{2T}\xi(t), \tag{25}$$

where $\mathrm{sgn}(x)$ denotes the signum function, i.e. $\mathrm{sgn}(-|x|) = -1$ and $\mathrm{sgn}(|x|) = +1$. The corresponding F-P equation reads

$$\partial_t \rho = \partial_v \left[\gamma \mathrm{sgn}(v)\rho + T\partial_v \rho\right]. \tag{26}$$

The normalized stationary state $\rho_s(v)$ is found as

$$\rho_s(v) = \frac{\gamma}{2T} e^{-\frac{\gamma|v|}{T}}. \tag{27}$$

We are interested in the stationary ACF $R(\tau)$ of the velocity of the particle, or, equivalently, in the power spectral density $S(\omega)$. To compute the ACF, we first find the eigenfunctions ϕ_n by solving the Schrödinger equation

$$\left[T\frac{\partial^2}{\partial x^2} - \left\{\frac{\gamma^2}{4T} - \gamma\delta(v)\right\}\right]\phi_n = -\lambda_n \phi_n, \tag{28}$$

where we have used the property of the signum function $(\mathrm{sgn}(x))' = 2\delta(x)$.

The spectrum of eigenvalues can be found from the last equation by setting $v \neq 0$

$$T\phi_n'' - \frac{\gamma^2}{4T}\phi_n = -\lambda_n \phi_n, \tag{29}$$

which yields

$$\phi_k(v) = C_1 \cos kv + C_2 \sin kv, \tag{30}$$

with

$$k = \sqrt{\left(\lambda_k - \frac{\gamma^2}{4T}\right)\frac{1}{T}}. \tag{31}$$

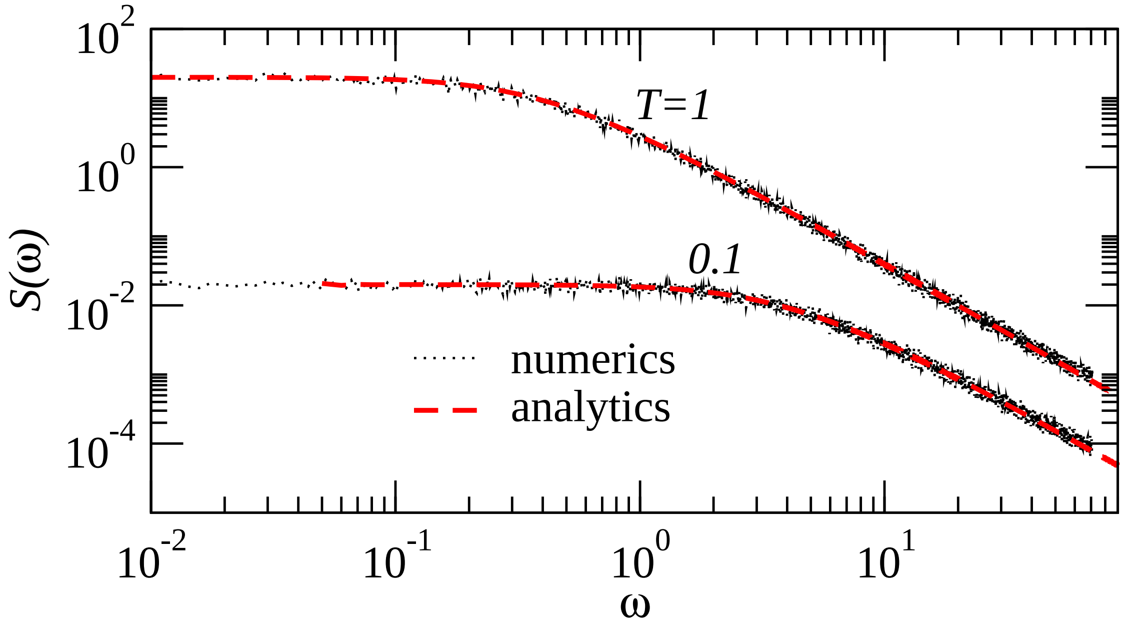

FIG. 1: Comparison of the analytical p.s.d. Eq. (37) with numerical simulations of the Langevin equation.

Consequently, for $\lambda_k \geq \frac{\gamma^2}{4T}$, the spectrum of eigenvalues is continuous and it is parametrized by $k > 0$

$$\lambda(k) = Tk^2 + \frac{\gamma^2}{4T}. \tag{32}$$

In order to compute the coefficients I_k in Eq. (23), one only requires the antisymmetric eigenfunctions ϕ_k. The later, normalized to a delta function, are given by

$$\phi_k = \frac{1}{\sqrt{\pi}} \sin kv. \tag{33}$$

Therefore

$$I_k \to I(k) = \int_{-\infty}^{\infty} dv\, v \left(\frac{1}{\sqrt{\pi}} \sin kv\right) \sqrt{\frac{\gamma}{2T}} e^{-\frac{-\gamma|v|}{2T}} = \sqrt{\frac{2\gamma}{\pi T}} \left[\frac{2\left(\frac{\gamma}{2T}\right) k}{(k^2 + (\frac{\gamma}{2T})^2)^2}\right]. \tag{34}$$

Because of the fact the eigenfunction $\rho_0(v)$, which corresponds to the zero eigenvalue is symmetric, the corresponding I_0 is zero. Additionally, we notice that because the spectrum is continuous, the stationary ACF is now given by the integral

$$G(\tau) = \int_0^{\infty} dk\, e^{-\lambda(k)\tau} I^2(k). \tag{35}$$

To simplify further calculations, we set without any loss of generality $\gamma = 2T$, implying that $\lambda(k) = T(1 + k^2)$.

With this, we obtain

$$G(\tau) = \int_0^\infty dk\, e^{-T(1+k^2)\tau} \frac{16}{\pi} \frac{k^2}{(1+k^2)^4}. \tag{36}$$

The last integral cannot be solved analytically. However, it is possible to obtain the analytic expression for the power spectrum

$$\begin{aligned} S(\omega) &= \frac{1}{\pi}\int_0^\infty e^{i\omega\tau} G(\tau) = \frac{16}{\pi^2}\int_0^\infty dk \frac{k^2}{(1+k^2)^4}\int_0^\infty d\tau e^{-T(1+k^2)\tau + i\omega\tau} \\ &= \frac{16}{\pi^2} Re\left\{\int_0^\infty dk\, \frac{k^2}{(1+k^2)^4}\frac{(-1)}{i\omega - T(1+k^2)}\right\} \\ &= \frac{16T}{\pi^2}\int_0^\infty \frac{k^2\, dk}{(1+k^2)^3(\omega^2 + T^2(1+k^2)^2)} = \frac{T^3}{\omega^4 \pi}\left[8 + x^2 - \frac{4(1+\sqrt{1+x^2})}{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1+x^2}}}\right], \end{aligned} \tag{37}$$

where $x = \omega/T$.

The analytical $S(\omega)$ Eq. (37) is compared with the results of numerical simulations in Fig. 1 for $T = 1$ and $T = 0.1$.

Linear response theory and Stochastic Resonance.

Consider a stochastic system, perturbed by a weak signal $f(t)$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\frac{dU(x)}{dx}\rho + f(t)\rho + T\frac{\partial \rho}{\partial x}\right). \tag{38}$$

The time-dependent solution of the last equation will generally be proportional to $f(t)$. In case when $f(t)$ is small, the linear response theory relates the solution of the time-dependent (perturbed) F-P equation to the ACF of the unperturbed (time-independent) equation. The essence of the linear response theory is the *Fluctuation-Dissipation Theorem* by *R. Kubo (1957)*.

Assume that $f(t)$ was constant $f(t) = f_0$ for $t < 0$ and that it has been switched off at $t = 0$, i.e. assume that

$$f(t) = f_0 \Theta(-t), \tag{39}$$

where $\Theta(x)$ is the step function.

The time-dependent average coordinate (also called the response of the system) is determined as

$$\langle x(t)|x_0, t_0 = 0\rangle = \int x\rho(x, t|x_0, t_0)\rho_s(x_0)\, dx\, dx_0. \tag{40}$$

The equilibrium $\rho_s(x_0)$ is Boltzmannian

$$\rho_s(x_0) = C \exp\left(-\frac{U(x) + x f_0}{T}\right). \tag{41}$$

For small f_0, we have

$$\rho_s(x_0) = C \exp\left(-\frac{U(x)}{T}\right)\left(1 - \frac{x f_0}{T} + O(f_0^2)\right). \tag{42}$$

Therefore

$$\langle x(t)|x_0, t_0\rangle = \int x\rho(x, t|x_0, t_0)\rho_s^{(0)}(x_0)\left[1 - \frac{x_0 f_0}{T}\right] dx\, dx_0, \tag{43}$$

where $\rho_s^{(0)}$ is the stationary distribution of the unperturbed system.

Taking into account that the force $f(t) = f_0\Theta(-t)$ is zero for $t > 0$, we conclude that $\rho(x, t|x_0, t_0)$ is the solution of the unperturbed equation. Therefore, we obtain

$$\langle x(t)|x_0, t_0\rangle = \langle x(t)\rangle_0 - \frac{f_0}{T}G(t - t_0), \tag{44}$$

with $\langle x(t)\rangle_0 = \int x\rho(x, t|x_0, t_0)\rho_s^{(0)}(x_0)\, dx\, dx_0$. On the other hand, we can express the response $\langle x(t)|x_0, t_0\rangle - \langle x(t)\rangle_0$ as the convolution of the time-dependent force and some kernel function ξ

$$\langle x(t)|x_0, t_0\rangle - \langle x(t)\rangle_0 = -f_0\int_0^\infty d\tau\, \chi(\tau)\Theta(\tau - t), \tag{45}$$

where the kernel function $\chi(x)$ is only defined for positive arguments

$$\chi(x) = 0, \quad x < 0. \tag{46}$$

Eq. (45) follows from the Green's function solution of the perturbed F-P equation

$$\begin{aligned}\langle x(t)|x_0, t_0\rangle - \langle x(t)\rangle_0 &= -f_0\int_{-\infty}^t d\tau\, \chi(t - \tau)\Theta(-\tau) = \{t - \tau = y\} \\ &= -f_0\int_0^\infty dy\, \chi(y)\Theta(y - t).\end{aligned} \tag{47}$$

By comparing Eq. (44) and Eq. (47), we obtain

$$\int_0^\infty dy\, \chi(y)\Theta(y - t) = \frac{1}{T}G(t - t_0) = \frac{1}{T}G(t). \tag{48}$$

Taking the derivative of the last equation w.r.t. $t - t_0 = t$, we finally get

$$\chi(t) = -\frac{1}{T}\frac{dG(t)}{dt}\Theta(t). \tag{49}$$

Introduce now the Fourier transform of $\chi(t)$

$$\hat{\chi}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(t)e^{-i\omega t}\,dt = \frac{1}{2\pi}\int_{0}^{\infty} \chi(t)e^{-i\omega t}\,dt \tag{50}$$

Noticing that $2\Theta(t)dG/dt = dG/dt + \mathrm{sgn}(t)dG/dt$ and taking the Fourier transform of Eq. (49), we arrive at Kubo's *Fluctuation-Dissipation Theorem*

$$\frac{2T}{\omega}\mathrm{Im}\hat{\chi}(\omega) = S(\omega), \tag{51}$$

where $S(\omega) = (1/2\pi)\int_{-\infty}^{\infty} G(\tau)e^{-i\omega\tau}\,d\tau$ is the stationary psd of the unperturbed system.

Periodic response

In case when the forcing is periodic i.e. if

$$f(t) = A\cos\Omega t, \tag{52}$$

we obtain

$$\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A\cos\Omega t e^{-i\omega t}\,dt = \frac{A}{2}\left(\delta(\omega-\Omega)+\delta(\omega+\Omega)\right). \tag{53}$$

Because the response of the system to perturbation $f(t)$ is given by the convolution of the kernel χ with $f(t)$

$$\delta x(t) \;=\; -\int_0^{\infty} dy\,\chi(y)f(y-t) = -\int_{-\infty}^{t} d\tau\,\chi(t-\tau)f(\tau) = -\int_{-\infty}^{+\infty} d\tau\,\chi(t-\tau)f(\tau), \tag{54}$$

we conclude by making use of the convolution theorem

$$\hat{\delta x}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta x(t)e^{-i\omega t}\,dt = -(2\pi)\hat{f}(\omega)\hat{\chi}(\omega). \tag{55}$$

Note that we use the following convention

$$\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt, \quad f(t) = \int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\,d\omega. \tag{56}$$

Therefore

$$\begin{aligned} -\frac{1}{2\pi}\delta x(t) \;&=\; \int_{-\infty}^{\infty}\hat{f}(\omega)\hat{\chi}(\omega)e^{i\omega t}\,d\omega = \int_{-\infty}^{\infty}\frac{A}{2}\left(\delta(\omega-\Omega)+\delta(\omega+\Omega)\right)\hat{\chi}(\omega)e^{i\omega t}\,dt \\ &=\; \frac{A}{2}\left[\hat{\chi}(\Omega)e^{i\Omega t}+\hat{\chi}(-\Omega)e^{-i\Omega t}\right] = \frac{A}{2}\left[\hat{\chi}(\Omega)e^{i\Omega t}+\hat{\chi}^*(\Omega)e^{-i\Omega t}\right] = A\mathrm{Re}\left[\hat{\chi}(\Omega)e^{i\Omega t}\right] \\ &=\; A[\hat{\chi}_1(\Omega)\cos\Omega t - \hat{\chi}_2(\Omega)\sin\Omega t] = A\sqrt{\hat{\chi}_1(\Omega)^2+\hat{\chi}_2(\Omega)^2}\cos(\Omega t-\phi), \end{aligned} \tag{57}$$

with real $\hat{\chi}_1$ and $\hat{\chi}_2$

$$\hat{\chi}(\omega) = \hat{\chi}_1(\omega) + i\hat{\chi}_2(\omega) \tag{58}$$

and the phase shift

$$\tan\phi(\Omega) = \frac{\hat{\chi}_2(\Omega)}{\hat{\chi}_1(\Omega)}. \tag{59}$$

Note that the *positive* amplitude of the forcing A corresponds to *negative* amplitude in the Langevin equation, i.e. by our convention we have

$$\dot{x} = -\frac{dU(x)}{dx} - A\cos\Omega t + \sqrt{2T}\xi(t), \tag{60}$$

which corresponds to

$$\frac{\partial\rho(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\frac{dU(x)}{dx}\rho + A\cos\Omega t\rho + T\frac{\partial\rho}{\partial x}\right]. \tag{61}$$

Stochastic Resonance

Discovered in 1982 by Roberto Benzi

Benzi R, Parisi G, Sutera A, Vulpiani A (1982). "Stochastic resonance in climatic change",

the main idea is that the response of a stochastic system to a periodic perturbation can be amplified by fluctuations.

Detailed theoretical study can be found in

Gammaitoni L, Hnggi P, Jung P, Marchesoni F (1998). "Stochastic resonance". Review of Modern Physics 70 (1): 22387.

Consider an overdamped Brownian particle, moving in a bistable potential $U(x)$

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2}. \tag{62}$$

The typical dependence of $x(t)$ for such a bistable system is shown in Fig. 2(a).

The power spectral density and the stationary ACF of the *unperturbed* bistable system has been derived earlier in connection with the *Random Telegraph Process*. It has been shown that the p.s.d. $S(\omega)$ is given by a Lorenzian

$$S(\omega) = \frac{2\lambda}{4\lambda^2 + \omega^2}, \tag{63}$$

where λ denotes the transition rate, which is symmetric for symmetric $U(x)$. The function $S(\omega)$ is shown in Fig. 2(b) and that the stationary ACF $G(\tau)$ decays exponentially

$$G(\tau) = e^{-2\lambda|\tau|}. \tag{64}$$

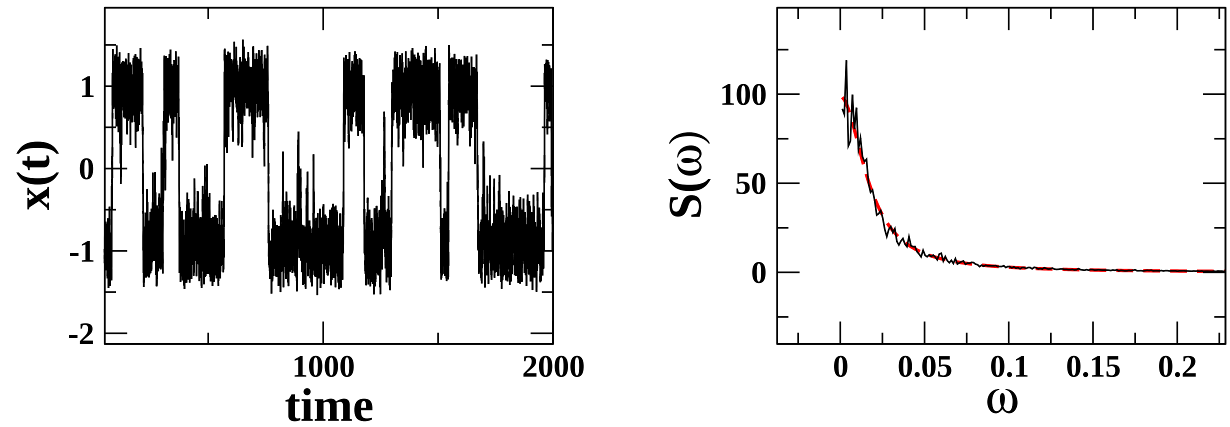

FIG. 2: (a) Typical time series of the particle trajectory in the bistable potential. (b) Power spectrum $S(\omega)$ of the unperturbed system.

We are interested in the amplitude of the linear response $\delta x(t)$ of such a bistable system to a weak periodic modulation $A\cos\Omega t$. First, we find the Fourier transform of the response function $\hat{\chi}(\omega)$

$$\hat{\chi}(\omega) = \frac{1}{2\pi}\int_0^\infty \left(\frac{2\lambda}{T}\right) e^{-i\omega\tau - 2\lambda\tau}\, d\tau = \left(\frac{\lambda}{\pi T}\right)\frac{2\lambda - i\omega}{\omega^2 + 4\lambda^2}. \tag{65}$$

Finally, using Eq. (57), we obtain

$$\delta x(t) = (2\pi)\frac{\lambda}{\pi T\sqrt{\omega^2+4\lambda^2}} A\cos(\Omega t - \phi) = A\frac{2\lambda}{T\sqrt{\omega^2+4\lambda^2}}\cos(\Omega t - \phi). \tag{66}$$

The transition rate λ can be found analytically for small values of the noise intensity T

$$\lambda(T) = \frac{1}{2\pi}\sqrt{-\frac{\partial^2 U(-1)}{\partial x^2}\frac{\partial^2 U(0)}{\partial x^2}}\exp\left(-\frac{\Delta U}{4T}\right), \tag{67}$$

where $\Delta U = 1/4$ is the height of the potential barrier.

The dependence of $|\delta x(t)|/A$ on the temperature T is shown in Fig. 3. It is remarkable that in both limits $T \to 0$ and $T \to \infty$, the response $\delta x(t)$ vanishes, attaining a maximum at a certain optimal $T_m(\Omega)$.

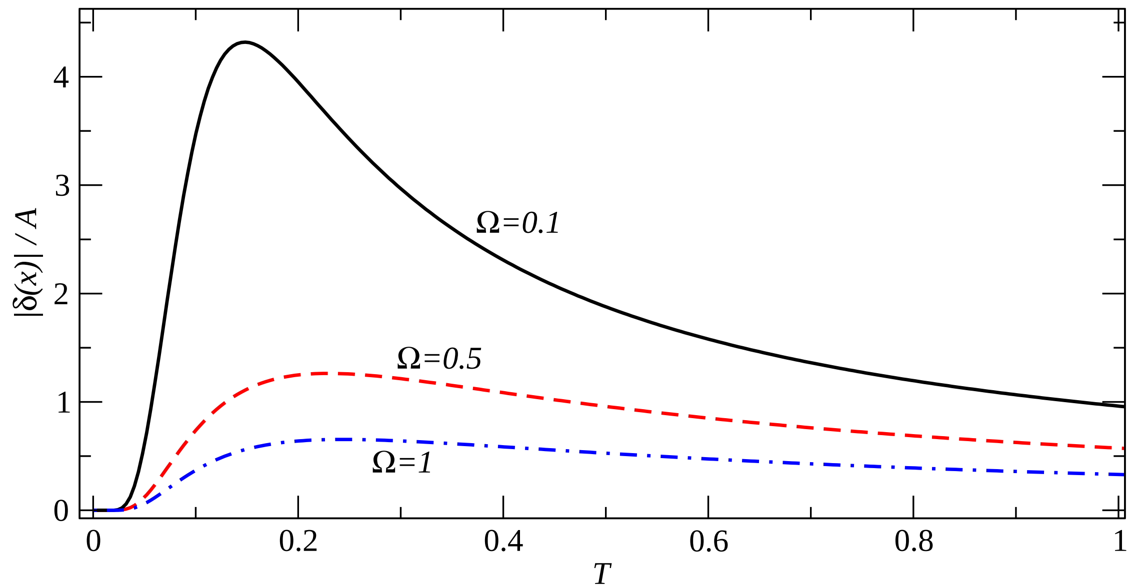

FIG. 3: The amplitude of the periodic response $|\delta x(t)|$ in units of A as a function of temperature T, as found from Eq. (66) for different values of the driving frequency Ω as given in the legend.