I. STOCHASTIC SYSTEMS WITH INTERACTIONS

Consider a system of N interacting overdamped particles. The equations of motion of such a system can be written as

$$\dot{x}_i = -\frac{dU(x_i)}{dx_i} + \sum_k C_{ik}(\{x_m\}) + \sqrt{2D}\xi_i(t), \tag{1}$$

where the matrix C_{ik} is often referred to as the coupling matrix.

In practice, two special kinds of coupling play central role: (i) the mean-field (all-to-all) coupling

$$C_{ik} = \frac{h(x_i)g(x_k)}{N}, \tag{2}$$

in which case, the Eq. (1) reduces to

$$\dot{x}_i = -\frac{dU(x_i)}{dx_i} + \frac{h(x_i)}{N}\sum_k g(x_k) + \sqrt{2D}\xi_i(t), \tag{3}$$

and the coupling via a pair interaction potential $W(\mid x_i - x_k \mid)$

$$C_{ik} = -\frac{\partial W(\mid x_i - x_k \mid)}{\partial x_i}, \quad C_{ii} = 0 \tag{4}$$

leading to

$$\dot{x}_i = -\frac{dU(x_i)}{dx_i} - \sum_{k \neq i} \frac{\partial W(\mid x_i - x_k \mid)}{\partial x_i} + \sqrt{2D}\xi_i(t). \tag{5}$$

The Fokker-Planck equation, which corresponds to Eq. (1), is written for the joint probability $\rho(x_1, x_2, x_3, \ldots, t)$

$$\frac{\partial \rho}{\partial t} = -\sum_i \frac{\partial J_i}{\partial x_i}, \tag{6}$$

with the probability current

$$J_i = -D\frac{\partial \rho}{\partial x_i} - \frac{dU(x_i)}{dx_i}\rho + \sum_k C_{ik}(\{x_m\})\rho, \tag{7}$$

and ρ normalized to one

$$\int \rho \, dx_1 \, dx_2 \, dx_3 \, \ldots \, dx_N = 1. \tag{8}$$

Remarkably, if the coupling matrix C_{ik} only depends on the coordinates of x_i and x_k, one can simplify the F-P equation by integrating it over $N-1$ coordinates and by assuming the natural boundary conditions. Thus, in case of the mean-field coupling, we obtain

$$\begin{aligned}\frac{\partial\rho_1}{\partial t} &= -\frac{\partial}{\partial x_1}\left(\int dx_N\, dx_{N-1}\, dx_{N-2}\,\ldots\, dx_2 J_1\right) = \\ &= \frac{\partial}{\partial x_1}\left(D\frac{\partial\rho_1}{\partial x_1}+\frac{dU(x_1)}{dx_1}\rho_1-\frac{h(x_1)g(x_1)}{N}\rho_1-\frac{h(x_1)}{N}\sum_{k=2}^{N}\int \rho_2(x_1,x_k)g(x_k)\,dx_k\right),\end{aligned} \tag{9}$$

where ρ_1 and ρ_2 are the one-body and the two-body density, respectively

$$\begin{aligned}\rho_1(x,t) &= \int dx_N\, dx_{N-1}\, dx_{N-2}\,\ldots\, dx_2\rho(x,x_2,\ldots,t), \\ \rho_2(x,y,t) &= \int dx_N\, dx_{N-1}\, dx_{N-2}\,\ldots\, dx_3\rho(x,y,x_3,\ldots,t).\end{aligned} \tag{10}$$

In order to decouple the equation for ρ_1 from ρ_2, one usually uses the so-called random-phase approximation (or mean-field approximation). To this end, one assumes that the conditional probability $\rho(x|y)$ is independent of y, in other words, we have

$$\rho_2(x,y,t) = \rho(x,t|y,t)\rho_1(y,t) = \rho_1(x,t)\rho_1(y,t). \tag{11}$$

With this assumption, we obtain a closed equation for ρ_1

$$\frac{\partial\rho_1}{\partial t} = \frac{\partial}{\partial x_1}\left(D\frac{\partial\rho_1}{\partial x_1}+\frac{dU(x_1)}{dx_1}\rho_1-\frac{h(x_1)g(x_1)}{N}\rho_1-(N-1)\rho_1\frac{h(x_1)}{N}\int g(y)\rho_1(y,t)\,dy\right). \tag{12}$$

In the thermodynamic limit $N\to\infty$, we get the nonlinear F-P equation in the mean-field approximation

$$\frac{\partial\rho_1}{\partial t} = \frac{\partial}{\partial x_1}\left(D\frac{\partial\rho_1}{\partial x_1}+\frac{dU(x_1)}{dx_1}\rho_1-\rho_1(x_1)h(x_1)\int g(y)\rho_1(y,t)\,dy\right), \tag{13}$$

with ρ_1 normalized to one, i.e.

$$\int \rho_1(x,t)\,dx = 1. \tag{14}$$

In case of the pair interaction, proceeding in a similar way, we obtain for finite N

$$\frac{\partial\rho_1}{\partial t} = \frac{\partial}{\partial x_1}\left(D\frac{\partial\rho_1}{\partial x_1}+\frac{dU(x_1)}{dx_1}\rho_1+(N-1)\rho_1(x,t)\int\frac{\partial W(\mid x-y\mid)}{\partial x}\rho_1(y,t)\,dy\right). \tag{15}$$

Multiplying both sides of the last equation with N and taking the limit $N\to\infty$, we get

$$\frac{\partial p_1}{\partial t} = \frac{\partial}{\partial x_1}\left(D\frac{\partial p_1}{\partial x_1}+\frac{dU(x_1)}{dx_1}p_1+p_1(x,t)\int\frac{\partial W(\mid x-y\mid)}{\partial x}p_1(y,t)\,dy\right), \tag{16}$$

where the new one-body density $p_1(x,t) = N\rho_1(x,t)$ is normalized to the total number of particles in the system

$$\int p_1(x,t)\,dx = N. \tag{17}$$

Bistable potential and ferromagnetic mean-field coupling

Assume that each particle moves in a symmetric bistable potential $U(x) = x^4/4 - x^2/2$ with two minima $x_m = \pm 1$, separated by a maximum at $x = 0$. Further on, set

$$h(x) = \alpha = cosnt, \quad g(x) = x. \tag{18}$$

The F-P equation for the one-body density in the mean-field approximation becomes

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial \rho}{\partial x} + \frac{dU(x)}{dx}\rho - \alpha\rho\langle x\rangle\right), \tag{19}$$

with

$$\langle x\rangle = \int_{-\infty}^{\infty} \rho(x,t)x\,dx, \quad \int_{-\infty}^{\infty} \rho(x,t)\,dx = 1. \tag{20}$$

In case when the coupling between the particles is attracting (ferromagnetic), increasing the attraction strength results in the ferromagnetic transition from the state with zero magnetization $\langle x\rangle = 0$ to a state with non-zero magnetization $\langle x\rangle \neq 0$.

We identify the system with an Ising model of a ferromagnetic material. Thus, each particle represents a single atom withe the spin either "up" (particle is at $x = +1$) or "down" (particle is at $x = -1$). In the disordered state, the average coordinate, identified with the magnetization is zero $\langle x\rangle = 0$. In the ordered state, the stationary density ρ_s is no longer symmetric, leading to a non-zero magnetization $\langle x\rangle \neq 0$.

In order to study this phase transition quantitatively, we formally solve the stationary F-P equation

$$\rho_s = C\exp\left(-\frac{U(x) - \alpha x\langle x\rangle}{D}\right). \tag{21}$$

Close to the point of the transition, the magnetization $\langle x\rangle$ is small. So that we can expand the density into a series

$$\rho_s = C\exp\left(-\frac{U(x)}{D}\right)\left(1 + \frac{\alpha x\langle x\rangle}{D} + \frac{1}{2}\left[\frac{\alpha x\langle x\rangle}{D}\right]^2 + \dots\right). \tag{22}$$

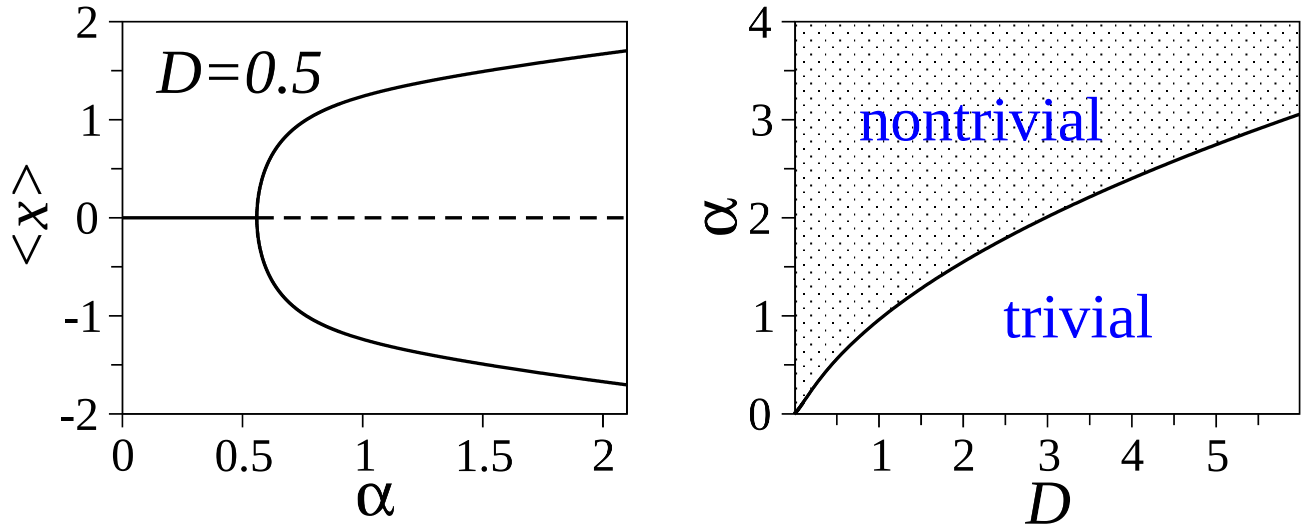

FIG. 1: (a) Pitchfork bifurcation of the order parameter $\langle x \rangle$ at fixed $D = 0.5$. (b) Phase diagram of the ordered-disordered states, as obtained from the condition Eq. (24).

Multiplying both sides of the last equation with x and integrating, we obtain

$$\langle x \rangle = \frac{\alpha}{D} \langle x \rangle \int_{-\infty}^{\infty} x^2 \rho_s^{(0)} \, dx, \tag{23}$$

with the symmetric unperturbed density $\rho_s^{(0)} = A \exp(-U/D)$. Finally, we obtain the condition for critical α_c and D_c, which mark the transition from the symmetric density to an asymmetric density

$$D_c \int_{-\infty}^{\infty} e^{-U(x)/D_c} \, dx = \alpha_c \int_{-\infty}^{\infty} x^2 e^{-U(x)/D_c} \, dx. \tag{24}$$

The typical bifurcation scenario of the order parameter $\langle x \rangle$ is shown in Fig. 1(a) at fixed $D = 0.5$. Increasing α, results in the pitchfork bifurcation of $\langle x \rangle$. There exist two possible non-trivial states: one with positive and one with negative $\langle x \rangle$. Fig. 1(b) shows the stability threshold $\alpha_c(D_c)$, as obtained from Eq. (24).

Relaxational dynamics and the H-theorem

M. Shiino "Dynamical behavior of stochastic systems of infinitely many coupled nonlinear oscillators exhibiting phase transitions of mean-field type: H theorem on asymptotic approach to equilibrium and critical slowing down of order-parameter fluctuations", Phys. Rev. A, **36** *2393 (1987)*

An important question is whether or not one could choose such coupling functions $h(x)$ and $g(x)$, so that the non-linear F-P equation will admit a stable time-periodic solution?

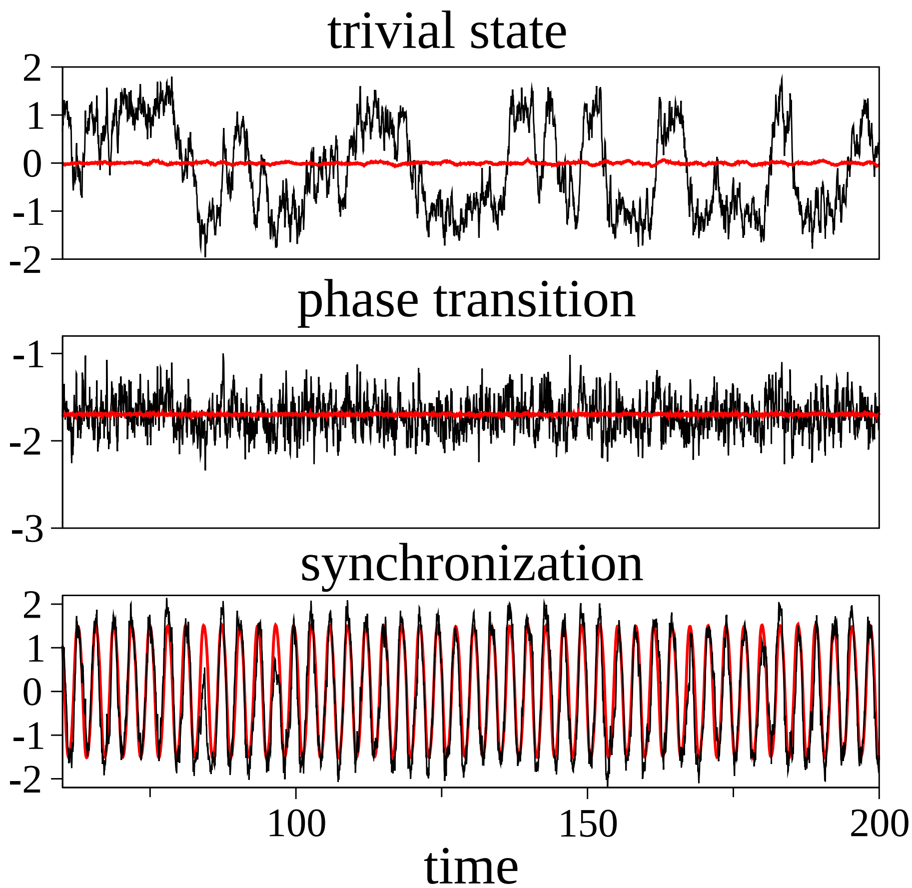

FIG. 2: Single trajectory (black) and the time-dependent mean field (red) in case of the symmetric state, phase transition and synchronization, from the top, respectively.

The answer is given by the so-called H-theorem by M.Shiino (1987) in case of linear coupling $g(x) = x$, $h(x) = const$. It is possible to show that the following functional

$$H[\rho] = \int \rho \ln\left(\frac{\rho}{q}\right) dx, \quad q = \exp\left(\frac{-U(x) + \alpha x \langle x \rangle}{D}\right), \tag{25}$$

when computed on the solutions of the time-dependent F-P equation, decreases monotonically with time, i.e.

$$\frac{dH[\rho]}{dt} \leq 0. \tag{26}$$

Consequently, stable time-periodic solutions are not possible.

Time-periodic global state: synchronization.

Interestingly, if the coupling is repulsive and contain time delay, the H-theorem is no longer applicable and one, generally, can expect to observe time-periodic states in the long-time limit.

The following simple system is due to:

[1] *D. Huber, L. S. Tsimring, Dynamics of an ensemble of noisy bistable elements with global time delayed coupling, Phys. Rev. Lett.* **91** *(2003) 260601,*

[2] *D. Huber, L. S. Tsimring, Cooperative dynamics in a network of stochastic elements with delayed feedback, Phys. Rev. E* **71** *(2005) 036150.*

Consider a repulsive (antiferromagnetic) mean field coupling with time delay

$$\dot{x}_i = -\frac{dU(x_i)}{dx_i} + \alpha\frac{1}{N}\sum_{j=1}^{N} g[x_j(t-\tau)] + \sqrt{2D}\xi_i(t), \tag{27}$$

with $\alpha < 0$ and some fixed delay time τ. The corresponding F-P equation in the mean-field approximation is given by

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}\rho - \alpha\bar{g}(t-\tau)\rho\right] + D\frac{\partial^2\rho}{\partial x^2}, \tag{28}$$

where

$$\bar{g}(t-\tau) = \int_{-\infty}^{\infty} g(x)\rho(x,t-\tau)\,dx. \tag{29}$$

The stationary one-particle distribution $\rho_0(x)$ obtained from Eq. (28) reads

$$\begin{aligned} \rho_0(x) &= \frac{1}{C}\exp\left[\frac{1}{D}\left(-U(x)+\alpha\bar{g}_0\right)\right], \\ C &= \int_{-\infty}^{\infty}\exp\left[\frac{1}{D}\left(-U(x)+\alpha\bar{g}_0\right)\right]dx, \end{aligned} \tag{30}$$

where stationary mean field $\bar{g}_0$ is computed self-consistently as $\bar{g}_0 = \int_{-\infty}^{\infty}\rho_0(x)g(x)\,dx$.

Transition to synchronized state as the Hopf bifurcation

The transition from the stationary to time-periodic density $\rho(x,t+2\pi/\omega) = \rho(x,t)$ is associated with the Hopf bifurcation of the mean field. Assume that near, but slightly above, synchronization threshold the mean field $\bar{g}(t-\tau)$ oscillates harmonically with a small amplitude ϵ and some (unknown) frequency ω around its stationary value, i.e. we can put $\bar{g}(t-\tau) = \bar{g}_0 + \epsilon\cos\omega t$. We have chosen the initial phase of these oscillations to be zero at the initial time moment. Using the ansatz

$$\rho(x,t) = \rho_0(x) + \epsilon\tilde{\rho}(x,t), \tag{31}$$

the linearized Fokker-Planck equation reads

$$\frac{\partial\tilde{\rho}}{\partial t} = L_0[\tilde{\rho}] - \frac{\partial}{\partial x}[\alpha(\bar{g}_0\tilde{\rho} + \cos\omega t\rho_0)], \tag{32}$$

where the stationary Fokker-Planck operator L_0 is given by

$$L_0 = \partial^2 U/\partial x^2 + (\partial U/\partial x)\partial/\partial x + D\partial^2/\partial x^2. \tag{33}$$

We look for the solution of Eq. (32) in the form

$$\tilde{\rho}(x,t) = u(x)\sin\omega t + v(x)\cos\omega t, \tag{34}$$

where $u(x)$ and $v(x)$ can be called linear response functions. Consequently, the mean field $\bar{g}(t)$ becomes

$$\bar{g}(t) \;=\; \int \rho(x,t)g(x)\,dx = \bar{g}_0 + \epsilon[\langle ug\rangle \sin\omega t + \langle vg\rangle \cos\omega t], \tag{35}$$

and $\bar{g}(t-\tau)$ is given by

$$\begin{aligned} \bar{g}(t-\tau) \;&=\; \bar{g}_0 + \epsilon[\langle ug\rangle \cos\omega\tau + \langle vg\rangle \sin\omega\tau]\sin\omega t \\ &+\; \epsilon[\langle vg\rangle \cos\omega\tau - \langle ug\rangle \sin\omega\tau]\cos\omega t \\ &=\; \bar{g}_0 + \epsilon\cos\omega t, \end{aligned} \tag{36}$$

where $\langle ug\rangle = \int_{-\infty}^{\infty} u(x)g(x)\,dx$ and $\langle vg\rangle = \int_{-\infty}^{\infty} v(x)g(x)\,dx$.

Substituting Eq. (34) into Eq. (32) we arrive at the following coupled equations for the unknown functions u, v and the onset frequency ω

$$\begin{aligned} \omega u \;&=\; L_0[v] - \alpha\frac{\partial}{\partial x}[(\bar{g}_0 v + \rho_0)], \\ -\omega v \;&=\; L_0[u] - \alpha\frac{\partial}{\partial x}[\bar{g}_0 u]. \end{aligned} \tag{37}$$

Additional coupling between the functions u and v is introduced from the self-consistent Eq. (36)

$$\begin{aligned} &\langle ug\rangle \cos\omega\tau + \langle vg\rangle \sin\omega\tau = 0, \\ &\langle vg\rangle \cos\omega\tau - \langle ug\rangle \sin\omega\tau = 1. \end{aligned} \tag{38}$$

Similar approach was used by N. Brunel in order to compute the location of the synchronization threshold for the ensemble of coupled leaky integrate-and-fire neurons:

[1] *N. Brunel, V. Hakim, M. J. E. Richardson, Firing-rate resonance in a generalized integrate-and-fire neuron with subthreshold resonance, Phys. Rev. E 67 (2003) 051916.*

[2] *N. Brunel, D. Hansel, How noise affects the synchronization properties of recurrent networks of inhibitory neurons, Neural Comp. 18 (2006) 10661110.*

Eigenfunction expansion: analytic results

The following analysis is as in:

A. Pototsky and N. Janson, "Synchronization of a large number of continuous one-dimensional stochastic elements with time delayed mean field coupling", Physica D, **238** *175-183 (2009)*

In what follows we study the transition from the trivial stationary state with

$$\bar{g}_0 = 0 \tag{39}$$

to a time-periodic state.

The boundary value problem Eqs. (37,38) can be formally solved by expanding the response functions u and v into a series of eigenfunctions of the stationary Fokker-Planck operator L_0.

Denote by ρ_n the n-th right-eigenfunctions of the operator L_0, i.e.

$$L_0\rho_n = -\lambda_n\rho_n, \tag{40}$$

where λ_n is the corresponding eigenvalue.

The Schrödinger equation for the functions

$$\phi_n(x) = \frac{\rho_n}{\rho_0(x)^{1/2}}, \tag{41}$$

is given by

$$H\phi_n(x) = -\lambda_n\phi_n(x), \tag{42}$$

with the Hamiltonian H

$$H = \left[\frac{1}{2}\left(\frac{\partial^2 U}{\partial x^2}\right) - \frac{1}{4D}\left(\frac{\partial U}{\partial x} - \alpha h\bar{g}_0\right)^2\right] + D\frac{\partial^2}{\partial x^2}. \tag{43}$$

Now we expand the response functions u and v into series of ρ_n,

$$u = \sum_{n=0}^{\infty} u_n\rho_n, \;\; v = \sum_{n=0}^{\infty} v_n\rho_n. \tag{44}$$

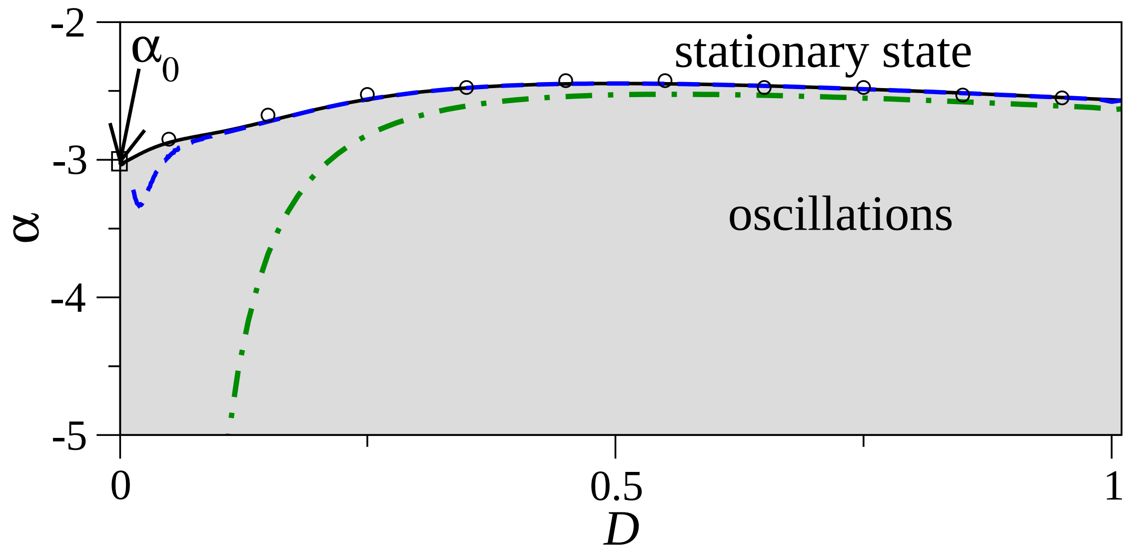

FIG. 3: Synchronization threshold in the parameter space (α, D) at fixed delay time $\tau = 1$ and $g(x) = x$.

The coefficients u_n and v_n of the expansions Eqs. (44) are determined from Eqs. (37) by multiplying both sides with ρ_n/ρ_0 and integrating over x from $-\infty$ to ∞, namely,

$$u_n = -\frac{\alpha\omega}{\lambda_n^2 + \omega^2}\left(\int_{-\infty}^{\infty} \frac{\rho_n}{\rho_0}\frac{\partial \rho_0}{\partial x}\,dx\right),$$
$$v_n = \frac{\lambda_n u_n}{\omega}. \tag{45}$$

From Eqs. (45) the mean field $\bar{g}$ is given by

$$\bar{g(t)} = -\alpha\epsilon \sum_{i=1}^{\infty} \frac{A_n \omega}{\lambda_n^2 + \omega^2}\sin\omega t - \alpha\epsilon \sum_{i=1}^{\infty} \frac{A_n \lambda_n}{\lambda_n^2 + \omega^2}\cos\omega t, \tag{46}$$

where the coefficients A_n read

$$A_n = \left(\int_{-\infty}^{\infty} g\rho_n\,dx\right)\left[\int_{-\infty}^{\infty} \frac{\rho_n}{\rho_0}\frac{\partial \rho_0}{\partial x}\,dx\right]. \tag{47}$$

From Eq. (46) and Eqs. (38) we obtain the following system of algebraic equations which determines the synchronization threshold for the coupling strength α together with the unknown onset frequency ω

$$\alpha \sum_{i=1}^{\infty} \frac{A_n \omega}{\lambda_n^2 + \omega^2} = \sin\omega\tau, \quad -\alpha \sum_{i=1}^{\infty} \frac{A_n \lambda_n}{\lambda_n^2 + \omega^2} = \cos\omega\tau. \tag{48}$$

If the coefficients A_n decrease rapidly with n, one can truncate the sums in Eqs. (48) after a few first terms. Truncation after the first term yields

$$\cos \omega\tau \; = \; -\frac{\alpha A_1 \lambda_1}{\sqrt{\lambda_1^2+\omega^2}}, \;\; 1 = \frac{|\alpha A_1|}{\sqrt{\lambda_1^2+\omega^2}}. \tag{49}$$

For $g(x) = x$ and small D, the coefficients A_1 and the eigenvalue λ_1 can be found analytically. In this case, we obtain

$$\cos(\omega\tau) \; = \; -\frac{2q_k}{\sqrt{4q_k^2+\omega^2}}, \;\; 1 = -\alpha\frac{\langle x^2\rangle_0}{D}\frac{2q_k}{\sqrt{4q_k^2+\omega^2}}, \tag{50}$$

where q_k stands for the Kramers transition rate.

These two equations determine the onset frequency ω together with a co-dimension one surface in the parameter space, for instance, one obtains a line $D(\alpha)$ for any fixed τ.

Fig. 3 shows the threshold in the parameter space (α, D) at fixed delay time $\tau = 1$ and $g(x) = x$. The solution of the boundary value problem Eqs. (37,38), is given by the solid line. In the shaded area below this line stable periodic solutions of the non-linear Fokker-Planck Eq. (28) exist, whereas above this line the trivial solution with $\bar{x}_0 = 0$ is stable. Dotted-dashed line was obtained by keeping only the first non-vanishing term A_1 in Eqs. (48). The dashed line corresponds to the first three non-vanishing terms A_n, $(n = 1, 3, 5)$ in Eqs. (48). Circles correspond to a direct simulation of the Langevin equations.