Swinburne University of Technology

Stochastic Differential Equations in Applications

TUTORIAL 2: Stratonovich and Ito calculus, the Fokker-Planck equation

---

- 1. The direction of swimming of an active particle is given by a unit vector $\vec{p}$, $|\vec{p}|= 1$. The equation of motion for $\vec{p}$ is given by a Stratonovich sde

$$\dot{\vec{p}} = \vec{\eta}(t) \times \vec{p},$$

where $\vec{\eta}(t) = (\eta_x(t), \eta_y(t), \eta_z(t))$ with $\langle \eta_i(t)\eta_k(t')\rangle = \delta_{ik}\delta(t - t')$.

You have decided to integrate this equation using the simple Euler scheme

$$\vec{p}(t + \delta t) = \vec{p}(t) + d\vec{W}(t) \times \vec{p}(t),$$

where $d\vec{W}(t)$ represents a vector-valued Wiener process.

Show that the magnitude of the vector $\vec{p}$ is not consereved in the Euler scheme and that the corresponding error is given by Error= $1/2 \mid \vec{p}(t) \mid \delta t$.

- 2. Let $\phi(t)$ be a one-dimensional Wiener process. In the class we have shown that the Stationary autocorrelation function for $q(t) = \cos[\phi(t)]$ is given by

$$\langle q(t)q(t')\rangle = \frac{1}{2}\exp\left(-\frac{1}{2}|t - t'|\right).$$

Confirm this result by the direct integration, taking into account the distribution density for the Wiener process

$$\rho(\phi, t|\phi_0 t_0) = \frac{1}{\sqrt{2\pi(t - t_0)}}\exp\left(-\frac{(\phi - \phi_0)^2}{2(t - t_0)}\right).$$

- 3. The evolution equation for the projection of the direction of swimming $q(t)$ of an active particle, which moves in 3D, is given by the Stratonovich sde

$$\dot{q} = -D_r q - \sqrt{1 - q^2}\eta(t),$$

with $\langle \eta(t)\eta(t')\rangle = 2D_r\delta(t - t')$. Using the corresponding Fokker-Planck equation, compute the conditional average $\langle q(t)|q_0, t_0\rangle$ and the stationary ACF $\langle q(t)q(t')\rangle$.