I. RECTIFIED BROWNIAN MOTION: BROWNIAN RATCHETS

The main idea is to create directed motion facilitated by thermal fluctuations.

Theoretical framework

Consider a (asymmetric) one-dimensional periodic potential $U(x)$ with period L, i.e. $U(x+L) = U(x)$. Typically, one uses the following biharmonic function $U(x) = \sin 2\pi x + (1/4)\sin 4\pi x$. The motion of a point particle m in $U(x)$ is then given by

$$\begin{aligned} \dot{x} &= v \\ m\dot{v} &= -\gamma v - U'(x) + F(t) + \sqrt{2\gamma kT}\xi(t), \end{aligned} \tag{1}$$

where $F(t)$ is some external generally time-dependent force and T stands for the temperature of thermal bath. In what follows, we assume that $F(t)$ is periodic with period T and unbiased, i.e. $\int_t^{t+T} F(t)\,dt = 0$.

In the overdamped limit, i.e. assuming that $m/\gamma \to 0$, it is possible to show that the dynamics of the particle is described by a single first-order equation

$$\dot{x} = -U'(x) + F(t) + \sqrt{2D}\xi(t). \tag{2}$$

The corresponding Fokker-Planck equation reads

$$\partial_t \rho = \partial_x[(U' - F)\rho + D\partial_x \rho] = -\partial j(x,t), \tag{3}$$

with the probability current density $j(x,t) = -(U' - F)\rho - D\partial_x\rho$.

Rocked ratchets: the adiabatic current

We are interested in the average current $\bar{J}$, given by

$$\bar{J} = \lim_{t\to\infty} \frac{1}{t}\int_x^{x+L} j(x',t)\,dx'\,dt. \tag{4}$$

Analytic results exist for the adiabatic limit, when $T \to \infty$.

In order to compute the average current in the adiabatic limit, we first determine the stationary current for a constant drive $F = const.$

The solution of the stationary F-P equation can be written as

$$\rho_s(x) = Ce^{(-U_{\text{eff}}(x)/D)} - \frac{J}{D}e^{(-U_{\text{eff}}(x)/D)}\int_0^x e^{(U_{\text{eff}}(y))/D}\,dy, \tag{5}$$

with the effective potential $U_{\text{eff}}(x) = U(x) - Fx$ ad two unknown constants C and J. Note that the constant J is in fact the stationary current.

The unknown constants can be found from the normalization condition on the density and from the fact that $\rho_s(x)$ is periodic with the period given by L.

The periodicity of $\rho_s(x)$ requires

$$\begin{aligned} & Ce^{(-U_{\text{eff}}(x+L)/D)} - \frac{J}{D} e^{(-U_{\text{eff}}(x+L)/D)} \int_0^{x+L} e^{(U_{\text{eff}}(y))/D}\, dy \\ = \ & Ce^{(-U_{\text{eff}}(x)/D)} - \frac{J}{D} e^{(-U_{\text{eff}}(x)/D)} \int_0^{x} e^{(U_{\text{eff}}(y))/D}\, dy \end{aligned} \tag{6}$$

Equivalently

$$C[e^{(FL/D)} - 1] - \frac{J}{D} e^{(FL/D)} [I(x+L) - I(0)] = -\frac{J}{D}[I(x) - I(0)], \tag{7}$$

where $I(x)$ denotes the indefinite integral $I(x) = \int e^{(U_{\text{eff}}(y))/D}\, dy$.

By differentiation, one can show that the function

$$g(x) = e^{(FL/D)} I(x+L) - I(x) \tag{8}$$

is in fact a constant, i.e. $g' = 0$. Therefore, without loss of generality, we can choose $x = 0$ in Eq. (7) and obtain

$$C = \frac{J e^{(FL/D)}[I(L) - I(0)]}{D[e^{(FL/D)} - 1]} \tag{9}$$

Finally, the stationary current J is found from the normalization condition

$$\int_x^{x+L} \rho(x')\, dx' = 1. \tag{10}$$

This condition can be written in terms of the two new functions

$$\begin{aligned} I^-(x) &= \int_0^x \exp\left(-U_{\text{eff}}(y)/D\right) dy \\ I^+(x) &= \int_0^x \exp\left(U_{\text{eff}}(y)/D\right) dy = I(x) - I(0) \end{aligned} \tag{11}$$

By setting $x = 0$ in Eq. (10), we obtain

$$CI^-(L) - \frac{J}{D} \int_0^L e^{(-U_{\text{eff}}(x)/D)} [I(x) - I(0)]\, dx = 1. \tag{12}$$

Consequently,

$$J = \frac{D(e^{(FL/D)} - 1)}{I^+(L) I^-(L) e^{(FL/D)} - (e^{(FL/D)} - 1) \int_0^L e^{(-U_{\text{eff}}(x)/D)} I^+(x)\, dx} \tag{13}$$

Typical dependence of J on F and D is shown in Fig. 1. Any asymmetric ratchet potential

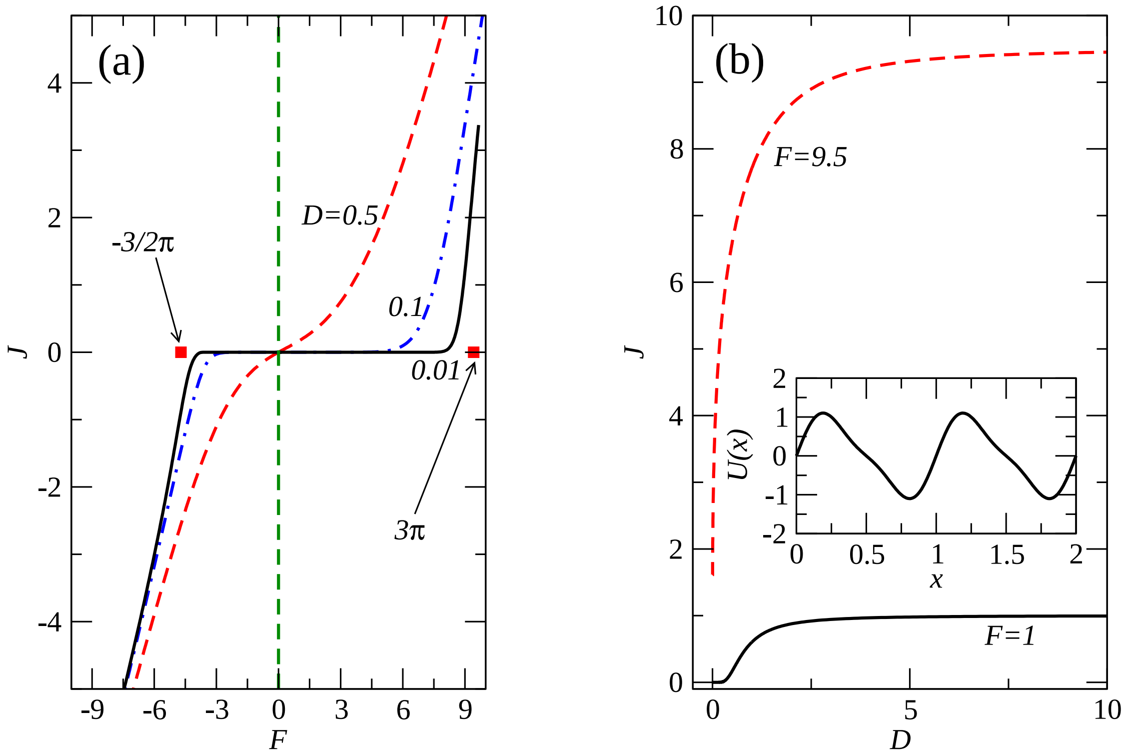

FIG. 1: (a) Stationary current J *vs* F for different values of D as in the legend. The left and right depinning forces $F_L = -3\pi/2$ and $F_R = 3\pi$ are shown by symbols on the horizontal axis. (b) J as a function of D for two different F as in the legend. Inset of (b): the ratchet potential $U(x) = \sin 2\pi x + 0.25 \sin 4\pi x$.

$U(x)$ is characterized by the left- and right depinning forces F_L and F_R, respectively. These determine the critical values of the external constant force, which is induces the unbounded motion of the particle in the negative and positive directions, respectively. For the standard biharmonic potential $U(x) = \sin 2\pi x + 0.25 \sin 4\pi x$, the depinning forces are given by $F_L = -3\pi/2$ and $F_R = 3\pi$, implying that the positive direction is the so-called "hard" direction (see the inset of Fig. 1(b)).

Consequently, in the limit of $D \to 0$, the current remains practically zero for $F \in [F_L, F_R]$, as shown in Fig. 1(a). If the noise intensity $D \neq 0$, the current is non-zero at any F. Important is that due to the asymmetry of the ratchet potential, the function $J(F)$ is *not an odd function*, implying that $J(F) \neq -J(-F)$. It is also worthwhile noticing that at any fixed F the current J approaches F as $D \to \infty$, as demonstrated in Fig. 1(b).

We are now ready to compute the adiabatic average current $\langle J \rangle$ for the unbiased time-

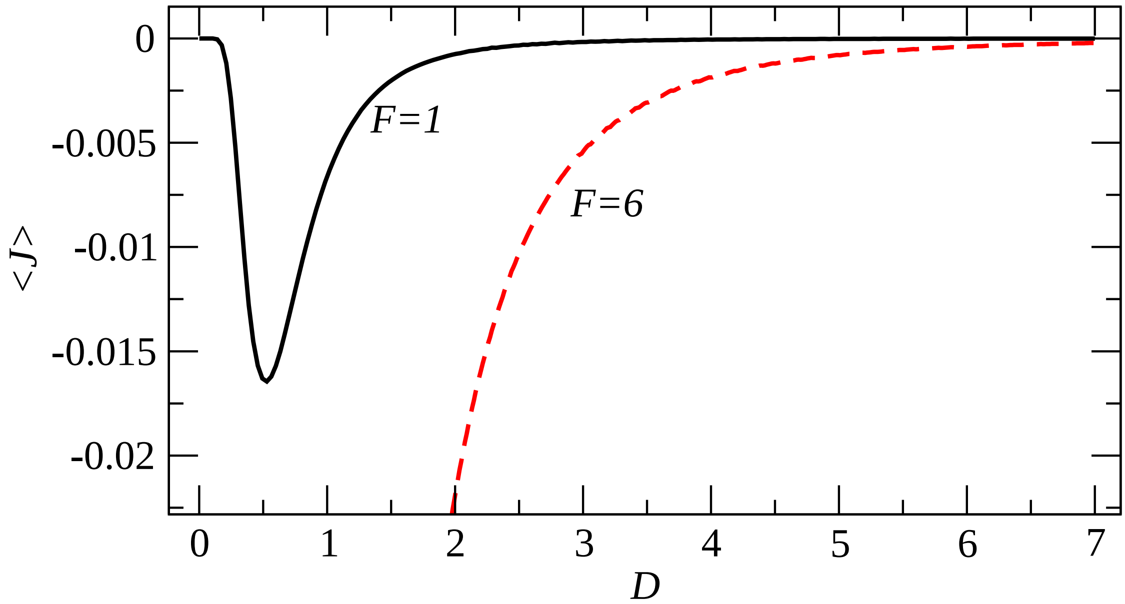

FIG. 2: Average adiabatic current $\langle J \rangle$ *vs* D for different values of the amplitude F as in the legend.

periodic drive $F(t)$ with vanishing frequency (infinitely large period). As such, we take the square-wave drive, which is given by $F(t) = \pm F$, where each value is kept over the half of the total period T.

Clearly, in this case, the average current is given by

$$\langle J \rangle = L \frac{J(F)T/2 + J(-F)T/2}{T} = L \frac{J(F) + J(-F)}{2}. \tag{14}$$

Typical dependence of the average adiabatic current $\langle J \rangle$ as a function of D is shown in Fig. 2. In the sub-critical regime, i.e. when $\mid F \mid < \mid F_L \mid$, the current is noise-induced and it disappears as $D \to \infty$.

Seebeck ratchet

After *M. Büttiker, Z. Phys. B* **68** *161 (1987) "Transport as a Consequence of State-Dependent Diffusion"*

Consider a Brownian particle in a periodic potential $U(x)$ in contact with a thermal bath with space-dependent temperature $T(x) = T(1 + w(x))^2$, where $w(x)$ is a periodic function such that $0 \leq 1 + w(x)$.

The Langevin equation is given by the Stratonovich sde

$$\dot{x} = -U'(x) + \sqrt{2T}[1 + w(x)]\xi(t), \tag{15}$$

where $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$.

The corresponding Fokker-Planck equation reads

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(U'(x)\rho + T(1+w(x))\frac{\partial}{\partial x}[(1+w(x))\rho]\right). \tag{16}$$

The stationary current J is given by

$$J = -U'(x)\rho - T(1+w(x))\frac{\partial}{\partial x}[(1+w(x))\rho]. \tag{17}$$

Rearranging terms in Eq. (17) we obtain

$$\frac{J}{(1+w(x))^2\rho} = -\frac{U'(x)}{(1+w(x))^2} - T\frac{w'(x)}{(1+w(x))} + T\frac{\rho'(x)}{\rho}. \tag{18}$$

By integrating Eq. (18) over one period with periodic boundary conditions, we obtain

$$\int_0^L \frac{J}{(1+w(x))^2\rho(x)}\,dx = -\int_0^L \frac{U'(x)}{(1+w(x))^2}\,dx. \tag{19}$$

Because the integral in the l.h.s. is always positive ($\rho(x) \geq 0$), we conclude that the stationary current J is non-zero only if $\int_0^L \frac{U'(x)}{(1+w(x))^2}\,dx \neq 0$. And conversely, if $\int_0^L \frac{U'(x)}{(1+w(x))^2}\,dx = 0$ then the stationary current vanishes, i.e. $J=0$.

Asymptotic expansion: limiting current at high temperature

Consider the limit of $T \to \infty$. In this case, the current $J(T)$ approaches a constant value, i.e. $\lim_{T\to\infty} J(T) = J_\infty = const$. In order to compute J_∞ we proceed as follows.

Rescale time according to

$$t' = T\,t. \tag{20}$$

In the rescaled units, the stationary current becomes

$$J' = -\frac{U'(x)\rho}{T} - (1+w(x))\frac{\partial}{\partial x}[(1+w(x))\rho]. \tag{21}$$

We are now looking for the solution of Eq. (21) in the form

$$\rho(x) = \rho_0(x) + \rho_1(x) + \ldots, \quad J' = J_0 + J_1 + \ldots, \tag{22}$$

where $\rho_i(x)$ and J_i are of the order of T^{-i}.

Then, in the zeroth order we obtain

$$J_0 = -(1+w(x))\frac{\partial}{\partial x}[(1+w(x))\rho_0(x)]. \tag{23}$$

The solution is given by $J_0 = 0$ and $\rho_0(x) = C_0/(1 + w(x))$, where $C_0^{-1} = \int_0^L dx/(1 + w(x))$, as obtained from the normalization condition $\int_0^L (\rho_0(x) + \rho_1(x) + \ldots)\, dx = 1$.

In the next order we have

$$J_1 = -U'(x)\rho_0(x) - (1 + w(x))\frac{\partial}{\partial x}[(1 + w(x))\rho_1(x)]. \tag{24}$$

The solution is given by

$$\rho_1(x) = \frac{1}{1 + w(x)} \int_0^x \left(-J_1 + \frac{U'(x)C_0}{1 + w(x)} \right) \frac{1}{1 + w(x)}\, dx + \frac{C_1}{1 + w(x)}. \tag{25}$$

The current J_1 is found from the periodicity condition for $\rho_1(x)$

$$\int_0^L \left(-J_1 + \frac{U'(x)C_0}{1 + w(x)} \right) \frac{1}{1 + w(x)}\, dx = 0 \tag{26}$$

Finally, going back to the original time $t = t'/T$, we get the average particle current $\langle \dot{x} \rangle$ in the limit of large temperature

$$\langle \dot{x} \rangle = L \lim_{T\to\infty} J(T) = -L \int_0^L \frac{U'(x)}{(1 + w(x))^2} \left[\int_0^L \frac{dx}{1 + w(x)} \right]^{-2}. \tag{27}$$

Thermal ratchets, the ratchet effect

Consider a so-called flashing or thermal ratchet, where the ratchet potential changes periodically in time (on and off ratchet). An overdamped Brownian particle will drift on average in the "hard" direction, even if a small external force is applied against this motion. This phenomenon is called the ratchet effect.

The fundamental characteristic of a thermal ratchet is the dependence of the average current $\langle J \rangle$ on the flashing frequency ω. It has been shown that in case when the external force is zero, the average particle current vanishes in both limits, when $\omega \to 0$ and $\omega \to \infty$ and reaches a maximum at a certain critical frequency ω_c.

Our aim is now to derive the asymptotic decay law for $\langle J \rangle$ in the limit of fast oscillations, i.e. when $\omega \to \infty$. To this end, we are following the original paper by

P. Reimann et al, "Brownian motors driven by temperature oscillations", Physics Letters A **215***, 26-31 (1996).*

Consider an overdamped particle in the ratchet potential $V(x)$ with time-modulated temperature of the thermal bath

$$T(t) = \bar{T} + \Delta(t), \tag{28}$$

with unbiased periodic modulation $\Delta(t+2\pi/\omega)=\Delta(t)$ and $\int_t^{t+2\pi/\omega}\Delta(t)\,dt=0$.

The Fokker-Planck equation is then given by

$$\partial_t\rho=\partial_x[V'(x)\rho+(\bar{T}+\Delta(t))\partial_x\rho]. \tag{29}$$

In order to study the limit $\omega\to\infty$, we rescale time according to $t'=t\omega$. The transformed F-P equation is

$$\partial_{t'}\rho=\frac{1}{\omega}\partial_x\left[V'(x)\rho+\bar{T}\partial_x\rho+\Delta\left(\frac{t'}{\omega}\right)\partial_x\rho\right]. \tag{30}$$

The rescaled modulation is normalized according to

$$\int_0^{2\pi}\Delta'(t')\,dt'=0. \tag{31}$$

In what follows we set for simplicity $t'\to t$.

Following standard methods of asymptotic analysis, we represent the time-dependent density ρ as an infinite series of the form

$$\rho(x,t)=\rho_0(x,t)+\frac{1}{\omega}\rho_1(x,t)+\frac{1}{\omega^2}\rho_2(x,t)+\dots, \tag{32}$$

where each function $\rho_i(x,t)$ is of the order of $O(\omega^0)$.

It is clear that this expansion implies the following normalization for ρ_i

$$\int_x^{x+L}\rho_i(x,t)\,dx=\delta_{i,0}. \tag{33}$$

Substituting the expansion Eq. (32) into the rescaled F-P equation Eq. (30), and comparing the terms of the same order of $(1/\omega)^i$, we obtain a sequence of coupled equations for $\rho_i(x,t)$. Thus, in the zeroth order, we get

$$\left(\frac{1}{\omega}\right)^0:\ \frac{\partial\rho_0}{\partial t}=0, \tag{34}$$

implying that $\rho_0=\rho_0(x)$. The next order gives

$$\left(\frac{1}{\omega}\right)^1:\ \frac{\partial\rho_1}{\partial t}=\frac{\partial}{\partial x}\left(V'\rho_0+\bar{T}\frac{\partial\rho_0}{\partial x}+\Delta(t)\frac{\partial\rho_0}{\partial x}\right). \tag{35}$$

By integrating Eq. (35) over one period of the forcing and taking into account that

$$\int_0^{2\pi}\frac{\partial\rho_1(x,t)}{\partial t}\,dt=0, \tag{36}$$

we obtain

$$0 = \frac{\partial}{\partial x}\left(V'\rho_0 + \bar{T}\frac{\partial \rho_0}{\partial x}\right), \tag{37}$$

and, consequently

$$\rho_0(x) = e^{\left(-\frac{V(x)}{T}\right)} / \left[\int_0^L e^{\left(-\frac{V(x)}{T}\right)}\, dx\right]. \tag{38}$$

From Eq. (35) we can now find ρ_1

$$\rho_1(x,t) = I(t)\rho_o'' + f_1(x), \tag{39}$$

where $I(t)$ is given by the indefinite integral

$$I(t) = \int \Delta(t)\, dt \tag{40}$$

and the function $f_1(x)$ is time-independent.

It is important to notice that by integrating the F-P equation Eq. (30) over one period of the forcing, and using the fact that $\int_0^{2\pi} \partial_t \rho(x,t)\, dt = 0$, we conclude that the time-averaged current density is x-independent, i.e.

$$\bar{J} = \frac{1}{2\pi}\int_0^{2\pi} j(x,t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} \left[-V'\rho - \bar{T}\partial_x\rho - \Delta(t)\partial_x\rho\right]\, dt = const. \tag{41}$$

This should hold in any order of $(1/\omega)^i$.

The current density in the order of $(1/\omega)^1$ is given by

$$j_1(x,t) = -V'\rho_1 - \bar{T}\partial_x\rho_1 - \Delta(t)\partial_x\rho_1. \tag{42}$$

In order to find the time-averaged density, we use the following result

$$\int \Delta(t)I(t)\, dt = \frac{I^2(t)}{2}. \tag{43}$$

Noticing that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \rho_1\, dt &= f_1(x), \\ \frac{1}{2\pi}\int_0^{2\pi} \Delta(t)\rho_1\, dt &= \rho_0''\frac{1}{2\pi}\int_0^{2\pi} \Delta(t)I(t)\, dt = \frac{\rho_0''}{2\pi}\left(\frac{I^2(2\pi)}{2} - \frac{I^2(0)}{2}\right) = 0, \end{aligned} \tag{44}$$

we obtain the time-averaged current in the order of $(1/\omega)^1$

$$\bar{J}_1 = -V'f_1(x) - \bar{T}\frac{\partial f_1(x)}{\partial x} = \mu_1 = const. \tag{45}$$

The solution of the last equation is

$$f_1(x) = Ce^{(-V(x)/\bar{T})} - \frac{\mu_1}{\bar{T}} e^{(-V(x)/\bar{T})} \int_0^x e^{(V(y)/\bar{T})}\, dy. \tag{46}$$

We need to impose two additional conditions on f_1, namely, the periodicity and the normalization conditions

$$f_1(x+L) = f_1(x), \quad \int_0^L f_1(x)\, dx = 0, \tag{47}$$

The periodicity implies

$$\frac{\mu_1}{\bar{T}} \int_x^{x+L} e^{(V(y)/\bar{T})}\, dy = 0, \tag{48}$$

which is only true if $\mu_1 = 0$. The normalization yields $C = 0$, and, consequently $f_1(x) = 0$.

As we see, the flux in the order of $(1/\omega)^1$, given by μ_1 is zero. Therefore, we proceed to the next order.

$$\left(\frac{1}{\omega}\right)^2 : \quad \frac{\partial \rho_2}{\partial t} = \frac{\partial}{\partial x}\left[V'\rho_1 + \bar{T}\frac{\partial \rho_1}{\partial x} + \Delta(t)\frac{\partial \rho_1}{\partial x}\right]. \tag{49}$$

Or, equivalently

$$\frac{\partial \rho_2}{\partial t} = \frac{\partial}{\partial x}\left[V'\rho_0'' I(t) + \bar{T} I(t)\rho_0''' + \Delta(t) I(t)\rho_0'''\right]. \tag{50}$$

This gives the expression for $\rho_2(x,t)$

$$\rho_2(x,t) = \frac{\partial}{\partial x}[V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}] I_2(t) + \frac{I^2(t)}{2}\rho_0^{(4)} + f_2(x), \tag{51}$$

where $f_2(x)$ depends only on x and the function $I_2(t)$ is given by the indefinite integral

$$I_2(t) = \int I(t)\, dt. \tag{52}$$

The time-averaged current in the order of $(1/\omega)^2$ is

$$\langle J\rangle_2 = -V'\langle\rho_2\rangle - \bar{T}\partial_x\langle\rho_2\rangle - \langle\Delta(t)\partial_x\rho_2\rangle, \tag{53}$$

where $\langle\ldots\rangle$ denotes the time-average.

In order to explicitly compute the time-average values, we make use of the following auxiliary results

$$\begin{aligned}
\langle\rho_2\rangle &= \rho_0^{(4)}\frac{\langle I^2(t)\rangle}{2} + f_2(x), \\
\langle\Delta(t)\rho_2\rangle &= \frac{\partial}{\partial x}\left(V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right)\frac{1}{2\pi}\int_0^{2\pi}\Delta(t) I_2(t)\, dt + \rho_0^{(4)}\frac{1}{2\pi}\int_0^{2\pi}\Delta(t)\frac{I^2(t)}{2}\, dt \\
&= -\frac{\partial}{\partial x}\left(V'\rho_o^{(2)} + \bar{T}\rho_0^{(3)}\right)\langle I^2(t)\rangle,
\end{aligned} \tag{54}$$

where we have used the fact that $\langle I(t)\rangle = (1/2\pi)[I_2(2\pi) - I_2(0)] = 0$ and that

$$\begin{aligned}
\int \Delta(t) I_2(t)\, dt &= I(t) I_2(t) - \int I^2(t)\, dt, \\
\frac{1}{2\pi}\int_0^{2\pi} \Delta(t) I_2(t)\, dt &= -\frac{1}{2\pi}\int_0^{2\pi} I^2(t)\, dt, \\
\int_0^{2\pi} \Delta(t)\frac{I^2(t)}{2}\, dt &= \frac{I^3(t)}{6}|_0^{2\pi} = 0.
\end{aligned} \quad (55)$$

Therefore, the time-averaged current becomes

$$\begin{aligned}
\bar{J}_2 &= -\left(V' f_2 + \bar{T}\frac{\partial f_2}{\partial x}\right) - \frac{\langle I^2(t)\rangle}{2}\left(V'\rho_0^{(4)} + \bar{T}\rho_0^{(5)}\right) \\
&+ \langle I^2(t)\rangle \frac{\partial^2}{\partial x^2}\left(V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right) = \mu_2 = const.
\end{aligned} \quad (56)$$

The last equation can be written in a slightly more compact form

$$\left(V' f_2 + \bar{T}\frac{\partial f_2}{\partial x}\right) = -\mu_2 + g(x), \quad (57)$$

with

$$g(x) = \frac{\langle I^2(t)\rangle}{2}\left(V'\rho_0^{(4)} + \bar{T}\rho_0^{(5)}\right) - \langle I^2(t)\rangle \frac{\partial^2}{\partial x^2}\left(V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right). \quad (58)$$

The solution for $f_2(x)$ is

$$f_2(x) = C e^{\left(-\frac{V(x)}{\bar{T}}\right)} + \frac{1}{\bar{T}} e^{\left(-\frac{V(x)}{\bar{T}}\right)} \int_0^x [g(y) - \mu_2] e^{\left(\frac{V(y)}{\bar{T}}\right)}\, dy. \quad (59)$$

From the periodicity of $f_2(x)$ we get

$$\mu_2 = \frac{1}{I^+}\int_x^{x+L} g(y) e^{\left(\frac{V(y)}{\bar{T}}\right)}\, dy, \quad (60)$$

with $I^+ = \int_x^{x+L} e^{\left(\frac{V(y)}{\bar{T}}\right)}\, dy$.

Explicitly, we have

$$\mu_2 = \frac{1}{I^+}\int_x^{x+L}\left[\frac{\langle I^2(t)\rangle}{2}\left(V'\rho_0^{(4)} + \bar{T}\rho_0^{(5)}\right) - \langle I^2(t)\rangle\frac{\partial^2}{\partial x^2}\left(V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right)\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx. \quad (61)$$

The first term in Eq. (61) in the square brackets can be computed using integration by parts

$$\int_x^{x+L}\left[V'\rho_0^{(4)} + \bar{T}\rho_0^{(5)}\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx = \int_x^{x+L}\left[V'\rho_0^{(4)} - \frac{1}{\bar{T}}\bar{T}\rho_0^{(4)}\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx = 0. \quad (62)$$

The second term in Eq. (61) in the square brackets is transformed by integrating it two times by parts

$$\int_x^{x+L}\frac{\partial^2}{\partial x^2}\left[V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx = \int_x^{x+L}\left(\frac{V''}{\bar{T}} + \frac{V'^2}{\bar{T}^2}\right)\left[V'\rho_0^{(2)} + \bar{T}\rho_0^{(3)}\right] e^{\left(\frac{V}{\bar{T}}\right)}\, dx. \quad (63)$$

Now recalling that

$$\begin{aligned}
\rho_0' &= -\left(\frac{1}{I^-}\right)\frac{V'}{\bar{T}}e^{\left(-\frac{V}{\bar{T}}\right)}, \\
\rho_0^{(2)} &= \left(\frac{1}{I^-}\right)\left(-\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)e^{\left(-\frac{V}{\bar{T}}\right)}, \\
\rho_0^{(3)} &= \left(\frac{1}{I^-}\right)\left(-\frac{V^{(3)}}{\bar{T}}+3\frac{V^{(2)}V^{(1)}}{\bar{T}^2}-\frac{V'^3}{\bar{T}^3}\right)e^{\left(-\frac{V}{\bar{T}}\right)},
\end{aligned} \tag{64}$$

with $I^- = \int_0^L e^{\left(-\frac{V(y)}{\bar{T}}\right)}\,dy$, we obtain

$$\int_x^{x+L}\left(\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)\left[V'\rho_0^{(2)}+\bar{T}\rho_0^{(3)}\right]e^{\left(\frac{V}{\bar{T}}\right)}\,dx \tag{65}$$

$$\begin{aligned}
&= \frac{1}{I^-}\int_x^{x+L}\left(\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)\left[V'\left(-\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)+\bar{T}\left(-\frac{V^{(3)}}{\bar{T}}+3\frac{V^{(2)}V^{(1)}}{\bar{T}^2}-\frac{V'^3}{\bar{T}^3}\right)\right]dx \\
&= \frac{1}{I^-}\int_x^{x+L}\left(\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)\left[-V^{(3)}+2\frac{V^{(2)}V^{(1)}}{\bar{T}}\right]dx.
\end{aligned} \tag{66}$$

The last integral can be simplified using the following results

$$\begin{aligned}
\int_x^{x+L} V''V^{(3)}\,dx &= \frac{V''^2}{2}\Big|_x^{x+L} = 0, \\
\int_x^{x+L} V'^3V^{(2)}\,dx &= \frac{V'^4}{4}\Big|_x^{x+L} = 0, \\
\int_x^{x+L} V'^2V^{(3)}\,dx &= -2\int_x^{x+L} V'V''^2\,dx.
\end{aligned} \tag{67}$$

This yields

$$\int_x^{x+L}\left(\frac{V''}{\bar{T}}+\frac{V'^2}{\bar{T}^2}\right)\left[V'\rho_0^{(2)}+\bar{T}\rho_0^{(3)}\right]e^{\left(\frac{V}{\bar{T}}\right)}\,dx = \frac{4}{\bar{T}^2I^-}\int_x^{x+L}dx\,V'V''^2. \tag{68}$$

Finally, we obtain the main result of our calculations, i.e. the average current in the second order of $(1/\omega)$

$$\mu_2 = -\frac{1}{\omega^2}\frac{2\int_0^{2\pi}I^2(t)\,dt}{\pi\bar{T}^2I^-I^+}\int_x^{x+L}dx\,V'V''^2. \tag{69}$$

II. EIGENFUNCTIONS OF THE FOKKER-PLANCK OPERATOR

Transformation to a Schrödinger equation

Consider the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\left[V'\rho + T\frac{\partial\rho}{\partial x}\right], \tag{70}$$

with arbitrary $V(x)$. The r.h.s. of the last equation can be written as in terms of the Fokker-Planck operator $\hat{L}$

$$\hat{L}\rho = [V'' + V'\partial_x + T\partial_x^2]\rho. \tag{71}$$

Note that the operator $\hat{L}$ is non-Hermitian, because of the first derivative ∂_x.

Recall that the momentum operator $-ih\partial_x$ is Hermitian.

Introduce a new function

$$\phi(x) = \frac{\rho(x)}{\sqrt{\rho_s(x)}}, \tag{72}$$

where $\rho_s(x)$ is the stationary solution of Eq. (70), given by

$$\rho_s(x) = Ce^{\left(-\frac{V(x)}{T}\right)}, \quad C = \left[\int_{-\infty}^{\infty} e^{\left(-\frac{V(x)}{T}\right)}\, dx\right]^{-1}. \tag{73}$$

Then we have

$$\begin{aligned} \rho' &= \left(\phi' - \frac{\phi V'}{2T}\right)\sqrt{C}e^{\left(-\frac{V(x)}{2T}\right)}, \\ \rho'' &= \left(\phi'' - \frac{\phi' V'}{T} - \frac{\phi V''}{2T} + \frac{\phi V'^2}{4T^2}\right)\sqrt{C}e^{\left(-\frac{V(x)}{2T}\right)}. \end{aligned} \tag{74}$$

Therefore, the Eq. (71) becomes

$$\hat{L}\rho = \left[\left(\frac{V''}{2} - \frac{V'^2}{4T}\right)\phi + T\phi''\right]\sqrt{\rho_s}, \tag{75}$$

or,

$$\hat{L}\rho = \sqrt{\rho_s}\hat{H}\phi, \tag{76}$$

where the new Hermitian operator $\hat{H}$ is given by

$$\hat{H} = \left(\frac{V''}{2} - \frac{V'^2}{4T}\right) + T\frac{\partial^2}{\partial x^2}, \tag{77}$$

Now by setting $\rho(x,t) = \phi(x,t)\sqrt{\rho_s(x)}$ in Eq. (70), we obtain the equation for ϕ

$$\frac{\partial \phi}{\partial t} = \hat{H}\phi. \tag{78}$$

Eigenfunction expansion

It is now convenient to represent the density $\rho(x,t)$ as a superposition of the eigenfunctions of $\hat{L}$

$$\rho(x,t|x_0,t_0) = \sum_0^\infty A_n(x_0)\rho_n(x)e^{-\lambda_n(t-t_0)}, \tag{79}$$

where index n numbers the eigenfunctions

$$\hat{L}\rho_n = -\lambda_n \rho_n. \tag{80}$$

Clearly, if $\rho_n = \sqrt{\rho_s}\phi_n$, then ϕ_n solves the eigenvalue problem of the operator $\hat{H}$ with *the same eigenvalues* λ_n

$$\hat{H}\phi_n = -\lambda_n \phi_n. \tag{81}$$

Because $\hat{H}$ is Hermitian, the eigenvalues λ_n are real and the eigenfunctions are orthogonal

$$\int_{-\infty}^{\infty} \phi_n(x)\phi_m(x)\,dx = \delta_{nm}, \tag{82}$$

where δ_{nm} should be replaced by $\delta(n-m)$ for continuous spectrum.

Consequently, the normalization of the functions ρ_n is given by

$$\int_{-\infty}^{\infty} \frac{\rho_n(x)\rho_m(x)}{\rho_s(x)}\,dx = \delta_{nm}, \tag{83}$$

where $\rho_0 = \rho_s$ and $\rho_n = \sqrt{\rho_s}\phi_n$, for $(n = 1, 2, 3, \ldots)$.

Positivity of eigenvalues

Consider the expression

$$\int_{-\infty}^{\infty} \frac{\rho_n(x)\hat{L}\rho_n(x)}{\rho_s(x)}\,dx = \int_{-\infty}^{\infty} \frac{\rho_n(x)(-\lambda_n\rho_n(x))}{\rho_s(x)}\,dx = -\lambda_n. \tag{84}$$

On the other hand, after integrating by parts, we have

$$\int_{-\infty}^{\infty} \frac{\rho_n(x)\hat{L}\rho_n(x)}{\rho_s(x)}\,dx = -\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\frac{\rho_n(x)}{\rho_s}\right)\left\{V'\rho_n + T\frac{\partial \rho_n}{\partial x}\right\}\,dx \tag{85}$$

But because

$$\frac{\partial}{\partial x}\left(\frac{\rho_n(x)}{\rho_s}\right) = \frac{1}{T\rho_s}\left(T\rho_n' + V'\rho_n\right), \tag{86}$$

we conclude that

$$\int_{-\infty}^{\infty} \frac{\rho_n(x)\hat{L}\rho_n(x)}{\rho_s(x)}\,dx = -\int_{-\infty}^{\infty} T\rho_s\left(\frac{\partial}{\partial x}\frac{\rho_n(x)}{\rho_s}\right)^2\,dx = -\lambda_n < 0. \tag{87}$$

Completeness

The completeness condition for the eigenfunctions ϕ_n can be expressed as

$$\sum_n \phi_n(x)\phi_n(x') = \delta(x - x'). \tag{88}$$

This condition can also be written in terms of ρ_n

$$\sum_n \frac{\rho_n(x)\rho_n(x')}{\sqrt{\rho_s(x')}\sqrt{\rho_s(x)}} = \frac{1}{\rho_s(x')}\sum_n \rho_n(x)\rho_n(x') = \delta(x - x'). \tag{89}$$

Transition probability density

Using the completeness condition, we can represent the time-dependent probability $\rho(x,t|x_0,t_0)$ in terms of ρ_n as follows

$$\begin{aligned} \rho(x,t|x_0,t_0) &= e^{\hat{L}(x)(t-t_0)}\delta(x - x_0) = e^{\hat{L}(x)(t-t_0)}\frac{1}{\rho_s(x_0)}\sum_n \rho_n(x)\rho_n(x_0) \\ &= \sum_n e^{-\lambda_n(t-t_0)}\frac{\rho_n(x_0)}{\rho_s(x_0)}\rho_n(x). \end{aligned} \tag{90}$$

Stationary ACF

By definition, the stationary ACF $R(\tau)$ is given by

$$\begin{aligned} R(\tau) &= \int dx\, dx'\, x\, x' \rho(x,t|x_0,t_0)\rho_s(x_0) \\ &= \int dx\, dx'\, x\, x' \sum_n e^{-\lambda_n(t-t_0)}\rho_n(x_0)\rho_n(x) = \sum_n e^{-\lambda_n(t-t_0)} I_n^2, \end{aligned} \tag{91}$$

with

$$I_n = \int_{-\infty}^{\infty} x\rho_n(x)\, dx. \tag{92}$$

Example: Brownian motion with dry friction

[1] de Gennes P-G 2005 "Brownian motion with dry friction" J. Stat. Phys. 119 95362 (2005)

followed by

[2] H. Touchette et al, "Brownian motion with dry friction: FokkerPlanck approach", J. Phys. A **43** *445002 (2010)*

Consider a Brownian particle moving at the presence of a dry-friction force, which is determined by

$$\boldsymbol{F}(\boldsymbol{v}) = \begin{cases} 0, & \boldsymbol{v} = 0 \\ -\gamma\frac{\boldsymbol{v}}{|\boldsymbol{v}|}, & \boldsymbol{v} \neq 0 \end{cases} \tag{93}$$

The 1D motion is then described by the Langevin equation

$$m\dot{v} = -\gamma \text{sgn}(v) + \sqrt{2T}\xi(t), \tag{94}$$

where sgn(x) denotes the signum function, i.e. $\text{sgn}(-|x|) = -1$ and $\text{sgn}(|x|) = +1$. The corresponding F-P equation reads

$$\partial_t \rho = \partial_v \left[\gamma \text{sgn}(v)\rho + T\partial_v \rho\right]. \tag{95}$$

The normalized stationary state $\rho_s(v)$ is found as

$$\rho_s(v) = \frac{\gamma}{2T} e^{-\frac{\gamma|v|}{T}}. \tag{96}$$

We are interested in the stationary ACF $R(\tau)$ of the velocity of the particle, or, equivalently, in the power spectral density $S(\omega)$. To compute the ACF, we first find the eigenfunctions ϕ_n by solving the Schrödinger equation

$$\left[T\frac{\partial^2}{\partial x^2} - \left\{\frac{\gamma^2}{4T} - \gamma\delta(v)\right\}\right]\phi_n = -\lambda_n \phi_n, \tag{97}$$

where we have used the property of the signum function $(\text{sgn}(x))' = 2\delta(x)$.

The spectrum of eigenvalues can be found from the last equation by setting $v \neq 0$

$$T\phi_n'' - \frac{\gamma^2}{4T}\phi_n = -\lambda_n \phi_n, \tag{98}$$

which yields

$$\phi_k(v) = C_1 \cos kv + C_2 \sin kv, \tag{99}$$

with

$$k = \sqrt{\left(\lambda_k - \frac{\gamma^2}{4T}\right)\frac{1}{T}}. \tag{100}$$

Consequently, for $\lambda_k \geq \frac{\gamma^2}{4T}$, the spectrum of eigenvalues is continuous and it is parametrized by $k > 0$

$$\lambda(k) = Tk^2 + \frac{\gamma^2}{4T}. \tag{101}$$

In order to compute the coefficients I_k in Eq. (92), one only requires the antisymmetric eigenfunctions ϕ_k. The later, normalized to a delta function, are given by

$$\phi_k = \frac{1}{\sqrt{\pi}} \sin kv. \tag{102}$$

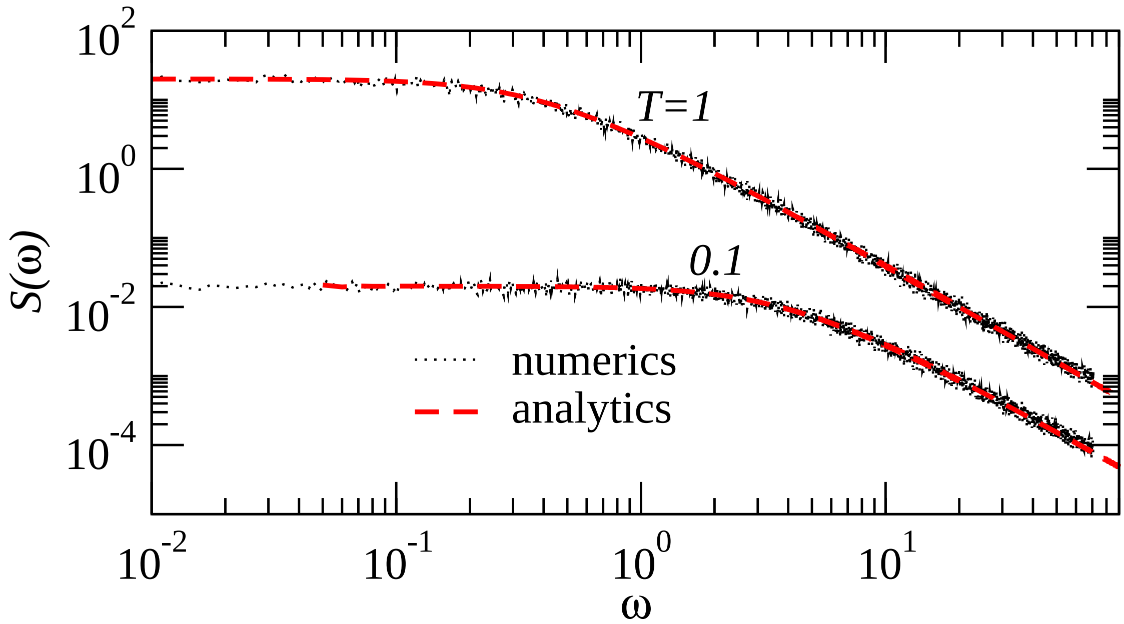

FIG. 3: Comparison of the analytical p.s.d. Eq. (106) with numerical simulations of the Langevin equation.

Therefore

$$I_k \to I(k) = \int_{-\infty}^{\infty} dv\, v \left(\frac{1}{\sqrt{\pi}} \sin kv\right) \sqrt{\frac{\gamma}{2T}} e^{-\frac{-\gamma|v|}{2T}} = \sqrt{\frac{2\gamma}{\pi T}} \left[\frac{2\left(\frac{\gamma}{2T}\right) k}{(k^2 + (\frac{\gamma}{2T})^2)^2}\right]. \tag{103}$$

Because of the fact the eigenfunction $\rho_0(v)$, which corresponds to the zero eigenvalue is symmetric, the corresponding I_0 is zero. Additionally, we notice that because the spectrum is continuous, the stationary ACF is now given by the integral

$$G(\tau) = \int_0^{\infty} dk\, e^{-\lambda(k)\tau} I^2(k). \tag{104}$$

To simplify further calculations, we set without any loss of generality $\gamma = 2T$, implying that $\lambda(k) = T(1 + k^2)$.

With this, we obtain

$$G(\tau) = \int_0^{\infty} dk\, e^{-T(1+k^2)\tau} \frac{16}{\pi} \frac{k^2}{(1 + k^2)^4}. \tag{105}$$

The last integral cannot be solved analytically. However, it is possible to obtain the analytic expression for the power spectrum

$$S(\omega) = \frac{1}{\pi} \int_0^{\infty} e^{i\omega\tau} G(\tau) = \frac{16}{\pi^2} \int_0^{\infty} dk \frac{k^2}{(1 + k^2)^4} \int_0^{\infty} d\tau e^{-T(1+k^2)\tau + i\omega\tau} \tag{106}$$

$$= \frac{16}{\pi^2} Re \left\{ \int_0^\infty dk \, \frac{k^2}{(1+k^2)^4} \frac{(-1)}{i\omega - T(1+k^2)} \right\}$$

$$= \frac{16T}{\pi^2} \int_0^\infty \frac{k^2 \, dk}{(1+k^2)^3(\omega^2 + T^2(1+k^2)^2)} = \frac{T^3}{\omega^4 \pi} \left[8 + x^2 - \frac{4(1+\sqrt{1+x^2})}{\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1+x^2}}} \right],$$

where $x = \omega/T$.

The analytical $S(\omega)$ Eq. (106) is compared with the results of numerical simulations in Fig. 3 for $T = 1$ and $T = 0.1$.

Linear response theory and Stochastic Resonance.

Consider a stochastic system, perturbed by a weak signal $f(t)$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left(\frac{dU(x)}{dx} \rho + f(t)\rho + T \frac{\partial \rho}{\partial x} \right). \tag{107}$$

The time-dependent solution of the last equation will generally be proportional to $f(t)$. In case when $f(t)$ is small, the linear response theory relates the solution of the time-dependent (perturbed) F-P equation to the ACF of the unperturbed (time-independent) equation. The essence of the linear response theory is the *Fluctuation-Dissipation Theorem* by *R. Kubo (1957)*.

Assume that $f(t)$ was constant $f(t) = f_0$ for $t < 0$ and that it has been switched off at $t = 0$, i.e. assume that

$$f(t) = f_0 \Theta(-t), \tag{108}$$

where $\Theta(x)$ is the step function.

The time-dependent average coordinate (also called the response of the system) is determined as

$$\langle x(t) | x_0, t_0 = 0 \rangle = \int x \rho(x, t | x_0, t_0) \rho_s(x_0) \, dx \, dx_0. \tag{109}$$

The equilibrium $\rho_s(x_0)$ is Boltzmannian

$$\rho_s(x_0) = C \exp \left(- \frac{U(x) + x f_0}{T} \right). \tag{110}$$

For small f_0, we have

$$\rho_s(x_0) = C \exp \left(- \frac{U(x)}{T} \right) \left(1 - \frac{x f_0}{T} + O(f_0^2) \right). \tag{111}$$

Therefore

$$\langle x(t)|x_0,t_0\rangle = \int x\rho(x,t|x_0,t_0)\rho_s^{(0)}(x_0)\left[1-\frac{x_0 f_0}{T}\right]dx\,dx_0, \tag{112}$$

where $\rho_s^{(0)}$ is the stationary distribution of the unperturbed system.

Taking into account that the force $f(t) = f_0\Theta(-t)$ is zero for $t > 0$, we conclude that $\rho(x,t|x_0,t_0)$ is the solution of the unperturbed equation. Therefore, we obtain

$$\langle x(t)|x_0,t_0\rangle = \langle x(t)\rangle_0 - \frac{f_0}{T}G(t-t_0), \tag{113}$$

with $\langle x(t)\rangle_0 = \int x\rho(x,t|x_0,t_0)\rho_s^{(0)}(x_0)\,dx\,dx_0$. On the other hand, we can express the response $\langle x(t)|x_0,t_0\rangle - \langle x(t)\rangle_0$ as the convolution of the time-dependent force and some kernel function ξ

$$\langle x(t)|x_0,t_0\rangle - \langle x(t)\rangle_0 = -f_0\int_0^\infty d\tau\,\chi(\tau)\Theta(\tau-t), \tag{114}$$

where the kernel function $\chi(x)$ is only defined for positive arguments

$$\chi(x) = 0, \quad x < 0. \tag{115}$$

Eq. (114) follows from the Green's function solution of the perturbed F-P equation

$$\begin{aligned}\langle x(t)|x_0,t_0\rangle - \langle x(t)\rangle_0 &= -f_0\int_{-\infty}^t d\tau\,\chi(t-\tau)\Theta(-\tau) = \{t-\tau = y\}\\ &= -f_0\int_0^\infty dy\,\chi(y)\Theta(y-t).\end{aligned} \tag{116}$$

By comparing Eq. (113) and Eq. (116), we obtain

$$\int_0^\infty dy\,\chi(y)\Theta(y-t) = \frac{1}{T}G(t-t_0) = \frac{1}{T}G(t). \tag{117}$$

Taking the derivative of the last equation w.r.t. $t - t_0 = t$, we finally get

$$\chi(t) = -\frac{1}{T}\frac{dG(t)}{dt}\Theta(t). \tag{118}$$

Introduce now the Fourier transform of $\chi(t)$

$$\hat{\chi}(\omega) = \frac{1}{2\pi}\int_{-\infty}^\infty \chi(t)e^{-i\omega t}\,dt = \frac{1}{2\pi}\int_0^\infty \chi(t)e^{-i\omega t}\,dt \tag{119}$$

Noticing that $2\Theta(t)dG/dt = dG/dt + \mathrm{sgn}(t)dG/dt$ and taking the Fourier transform of Eq. (118), we arrive at Kubo's *Fluctuation-Dissipation Theorem*

$$\frac{2T}{\omega}\mathrm{Im}\hat{\chi}(\omega) = S(\omega), \tag{120}$$

where $S(\omega) = (1/2\pi)\int_{-\infty}^{\infty} G(\tau)e^{-i\omega\tau}\, d\tau$ is the stationary psd of the unperturbed system.

Periodic response

In case when the forcing is periodic i.e. if

$$f(t) = A\cos\Omega t, \tag{121}$$

we obtain

$$\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A\cos\Omega t e^{-i\omega t}\, dt = \frac{A}{2}\left(\delta(\omega-\Omega)+\delta(\omega+\Omega)\right). \tag{122}$$

Because the response of the system to perturbation $f(t)$ is given by the convolution of the kernel χ with $f(t)$

$$\delta x(t) \;=\; -\int_{0}^{\infty} dy\, \chi(y) f(y-t) = -\int_{-\infty}^{t} d\tau\, \chi(t-\tau) f(\tau) = -\int_{-\infty}^{+\infty} d\tau\, \chi(t-\tau) f(\tau), \tag{123}$$

we conclude by making use of the convolution theorem

$$\hat{\delta x}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta x(t) e^{-i\omega t}\, dt = -(2\pi)\hat{f}(\omega)\hat{\chi}(\omega). \tag{124}$$

Note that we use the following convention

$$\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt, \quad f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\, d\omega. \tag{125}$$

Therefore

$$\begin{aligned}
-\frac{1}{2\pi}\delta x(t) \;&=\; \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{\chi}(\omega)e^{i\omega t}\, d\omega = \int_{-\infty}^{\infty} \frac{A}{2}\left(\delta(\omega-\Omega)+\delta(\omega+\Omega)\right)\hat{\chi}(\omega)e^{i\omega t}\, dt \\
&=\; \frac{A}{2}\left[\hat{\chi}(\Omega)e^{i\Omega t} + \hat{\chi}(-\Omega)e^{-i\Omega t}\right] = \frac{A}{2}\left[\hat{\chi}(\Omega)e^{i\Omega t} + \hat{\chi}^{*}(\Omega)e^{-i\Omega t}\right] = A\mathrm{Re}\left[\hat{\chi}(\Omega)e^{i\Omega t}\right] \\
&=\; A[\hat{\chi}_1(\Omega)\cos\Omega t - \hat{\chi}_2(\Omega)\sin\Omega t] = A\sqrt{\hat{\chi}_1(\Omega)^2+\hat{\chi}_2(\Omega)^2}\cos\left(\Omega t-\phi\right),
\end{aligned} \tag{126}$$

with real $\hat{\chi}_1$ and $\hat{\chi}_2$

$$\hat{\chi}(\omega) = \hat{\chi}_1(\omega) + i\hat{\chi}_2(\omega) \tag{127}$$

and the phase shift

$$\tan\phi(\Omega) = \frac{\hat{\chi}_2(\Omega)}{\hat{\chi}_1(\Omega)}. \tag{128}$$

Note that the *positive* amplitude of the forcing A corresponds to *negative* amplitude in the Langevin equation, i.e. by our convention we have

$$\dot{x} = -\frac{dU(x)}{dx} - A\cos\Omega t + \sqrt{2T}\xi(t), \tag{129}$$

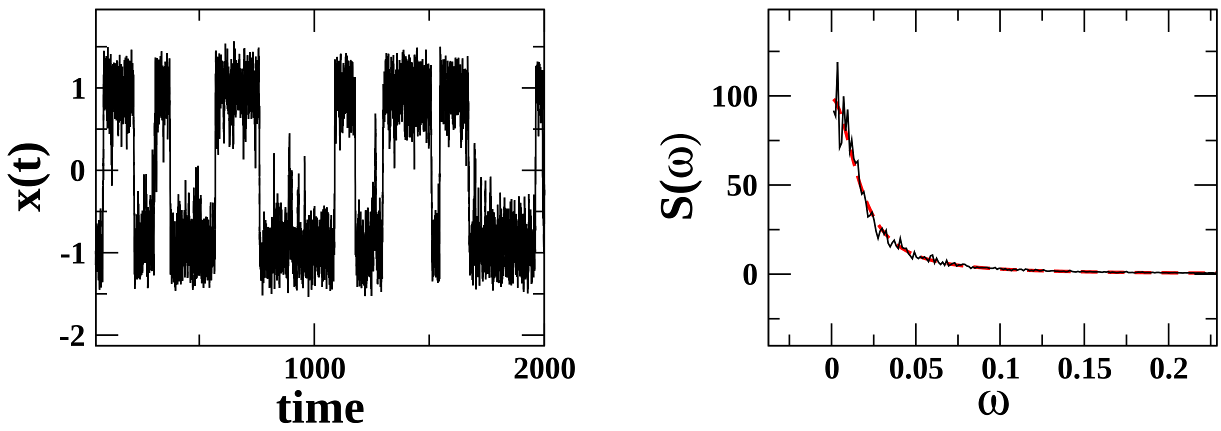

FIG. 4: (a) Typical time series of the particle trajectory in the bistable potential. (b) Power spectrum $S(\omega)$ of the unperturbed system.

which corresponds to

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\frac{dU(x)}{dx}\rho + A\cos\Omega t\rho + T\frac{\partial \rho}{\partial x}\right]. \tag{130}$$

Stochastic Resonance

Discovered in 1982 by Roberto Benzi

Benzi R, Parisi G, Sutera A, Vulpiani A (1982). "Stochastic resonance in climatic change",

the main idea is that the response of a stochastic system to a periodic perturbation can be amplified by fluctuations.

Detailed theoretical study can be found in

Gammaitoni L, Hnggi P, Jung P, Marchesoni F (1998). "Stochastic resonance". Review of Modern Physics 70 (1): 22387.

Consider an overdamped Brownian particle, moving in a bistable potential $U(x)$

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2}. \tag{131}$$

The typical dependence of $x(t)$ for such a bistable system is shown in Fig. 4(a).

The power spectral density and the stationary ACF of the *unperturbed* bistable system has been derived earlier in connection with the *Random Telegraph Process*. It has been shown that the p.s.d. $S(\omega)$ is given by a Lorenzian

$$S(\omega) = \frac{2\lambda}{4\lambda^2 + \omega^2}, \tag{132}$$

where λ denotes the transition rate, which is symmetric for symmetric $U(x)$. The function $S(\omega)$ is shown in Fig. 4(b) and that the stationary ACF $G(\tau)$ decays exponentially

$$G(\tau) = e^{-2\lambda|\tau|}. \tag{133}$$

We are interested in the amplitude of the linear response $\delta x(t)$ of such a bistable system to a weak periodic modulation $A \cos \Omega t$. First, we find the Fourier transform of the response function $\hat{\chi}(\omega)$

$$\hat{\chi}(\omega) = \frac{1}{2\pi} \int_0^\infty \left(\frac{2\lambda}{T}\right) e^{-i\omega\tau - 2\lambda\tau}\, d\tau = \left(\frac{\lambda}{\pi T}\right) \frac{2\lambda - i\omega}{\omega^2 + 4\lambda^2}. \tag{134}$$

Finally, using Eq. (126), we obtain

$$\delta x(t) = (2\pi) \frac{\lambda}{\pi T \sqrt{\omega^2 + 4\lambda^2}} A \cos(\Omega t - \phi) = A \frac{2\lambda}{T\sqrt{\omega^2 + 4\lambda^2}} \cos(\Omega t - \phi). \tag{135}$$

The transition rate λ can be found analytically for small values of the noise intensity T

$$\lambda(T) = \frac{1}{2\pi} \sqrt{-\frac{\partial^2 U(-1)}{\partial x^2} \frac{\partial^2 U(0)}{\partial x^2}} \exp\left(-\frac{\Delta U}{4T}\right), \tag{136}$$

where $\Delta U = 1/4$ is the height of the potential barrier.

The dependence of $|\delta x(t)|/A$ on the temperature T is shown in Fig. 5. It is remarkable that in both limits $T \to 0$ and $T \to \infty$, the response $\delta x(t)$ vanishes, attaining a maximum at a certain optimal $T_m(\Omega)$.

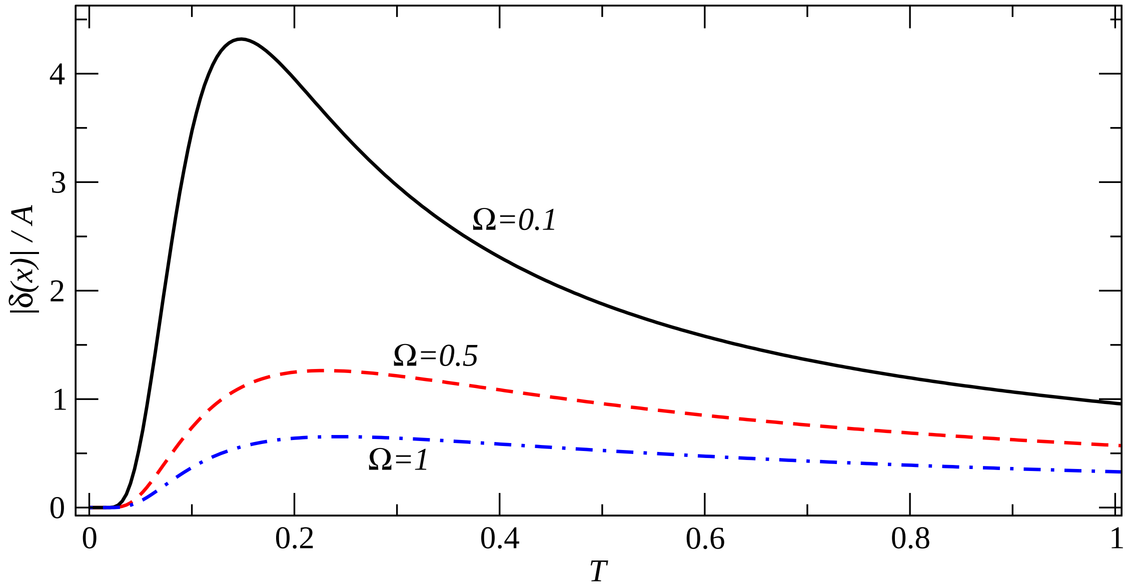

FIG. 5: The amplitude of the periodic response $|\delta x(t)|$ in units of A as a function of temperature T, as found from Eq. (135) for different values of the driving frequency Ω as given in the legend.