We consider Langevin equations:

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -x(x, v) - \frac{\partial U(x)}{\partial x} + \sqrt{2D} \xi(t)
\end{align*}
\]

\(x\) and \(v\) are the position and the velocity of a particle which moves in the potential \(U(x)\) with friction \(\xi(x, v)\). If \(\xi = \text{const}\) we deal with damped oscillations with a nonlinear potential (non-conservative).

If the damping term is nonlinear we have a dissipative nonlinear system with \(U\) being a harmonic potential in most cases.

The FPE for the transition probability density \(p_2 = p_2(x, v, t | x_0, v_0, t_0)\):

\[
\frac{\partial}{\partial t} p_2 + v \frac{\partial}{\partial x} p_2 - \frac{\partial U}{\partial x} \frac{\partial}{\partial v} p_2 = \frac{\partial}{\partial v} \xi(x, v) v p_2 + \xi \frac{\partial^2}{\partial v^2} p_2
\]

In the stationary limit \(p_2\) tends to its stationary density, \(p_2 \to p^s(x, v)\) with \(\frac{\partial p^s}{\partial t} = 0\) and does not depend on the initial state.
Amplitude and phase description

An analytical approach for the case of harmonic potential and weak dissipative forces. The main idea:

We search for solutions of the deterministic problem by transforming to an amplitude and phase description.

Small dissipation \(\Rightarrow\) one may assume a slow variation of the amplitude of oscillations \(A\) and a shift \(\phi\) of the phase \(\phi = \phi_0 + \phi\) added to the fast motion \(\phi_0 = \omega_0 t\), where \(\omega_0\) is the natural frequency of the oscillator.

Let us consider the harmonic potential \(U(x) = \omega_0^2 x^2/2\) and the following transformation of variables:

\[ x = A \cos (\omega_0 t + \phi), \quad \xi = -\omega_0 A \sin (\omega_0 t + \phi) \]

The Langevin equations for the amplitude \(A\) and the phase shift \(\phi\) are

\[ \dot{A} = \frac{\sin(\omega_0 t + \phi)}{\omega_0} \xi(x, \xi) + y_A, \]
\[ \dot{\phi} = \frac{\cos(\omega_0 t + \phi)}{A \omega_0} \xi(x, \xi) + y_\phi, \]

where \(y_A, y_\phi\) are new noise sources:

\[ y_A = -\frac{\sin(\omega_0 t + \phi)}{\omega_0} \sqrt{2 \xi \xi(1/4)}, \quad y_\phi = \frac{\cos(\omega_0 t + \phi)}{A \omega_0} \sqrt{2 \xi \xi(1/4)} \]
The first term in both equations for the amplitude and phase can be treated by averaging over one period $T = 2\pi/\omega_0$ under the assumption of constant $A$ and $\phi$. The noise in the equations for $A$ and $\phi$ is multiplicative.

This procedure gives the first-order expansions for $(a, \phi)$ and $(\omega, \phi)$ of the considered theory.

In the first kinetic coefficient one obtains from the noise terms

$$K^A_1 \sim D \frac{\cos^2(\omega_0 t + \phi)}{A\omega_0^2}, \quad K^\phi_1 \sim -D \frac{\cos(\omega_0 t + \phi) \sin(\omega_0 t + \phi)}{A^2 \omega_0^2}$$

After averaging over one period only one nonvanishing contribution $\frac{D}{2A\omega_0^2}$ remains in $K^A_1$. The second (kinetic) moments can be calculated following Eq. (5.2) [see lecture 5] and after averaging over one period the cross-correlations vanish: $\Rightarrow \Rightarrow K^{A,\phi}_2 = 0$. The noise sources of the amplitude and phase shift are, therefore, independent Gaussian white noise intensities:

$$K^{A,\phi}_2 = \frac{D}{2\omega_0^2}, \quad K^{\phi,\phi}_2 = \frac{D}{2\omega_0^2 A^2}.$$ 

Thus, the FPE in the amplitude-phase variables for the transition probability density $p_2(A, \phi, \omega | A_0, \phi_0, \omega_0)$:
\[
\frac{\partial}{\partial t} P_2 = -\frac{\partial}{\partial A} \left( f_A + \frac{\partial}{2 A \omega_0^2} \right) P_2 - \frac{\partial}{\partial y} f_y P_2 + \\
+ \frac{\partial}{2 \omega_0^2} \frac{\partial^2 P_2}{\partial A^2} + \frac{\partial}{2 \omega_0^2 A^2} \frac{\partial^2 P_2}{\partial y^2}
\]

The Rayleigh distribution solves in the stationary case this equation with \( \delta = \text{const} \) and a harmonic potential with frequency \( \omega_0 \).

\[
P^s(A, y) = \frac{1}{2\pi} \frac{A}{\sigma^2} \exp \left( -\frac{A^2}{2\sigma^2} \right),
\]

\[
\delta^2 = \frac{\delta}{\omega_0^2 \delta}.
\]

Energy Representation

An important representation of FPE uses energy as a variable. This representation is preferable for the cases of slow energy variations.

Let the friction be linear: \( \gamma = \text{const.} \). Then the velocity can be replaced by the energy as:

\[
\sigma(x, E) = \pm \sqrt{2 \left[ E - U(x) \right]},
\]

using the definition of mechanical energy. As a result we obtain Langevin equations (after substituting \( \dot{\gamma} \) into the eq-s):

\[
\dot{x} = \sigma(x, E), \quad \dot{E} = -\gamma \dot{x}^2 / \sigma(x, E) + \sqrt{2 \gamma} \sigma(x, E) \xi(t)
\]

The equation for the energy contains multiplicative noise \( \gamma \xi(x, E) \xi(t) \rightarrow \text{source of fluctuations in this representation.} \)

Kinetic coefficients can be found:

\[
K^x_1 = \sigma(x, E), \quad K^E_1 = -\gamma \dot{x}^2 / \sigma(x, E) + \gamma, \quad K^E_2 = \gamma \dot{x}^2 / \sigma(x, E)
\]

Thus, the FPE in a coordinate and energy representation for the transition probability density \( p^s(x, E, t | x_0, E_0, t_0) \) is

\[
\frac{\partial p^s}{\partial t} = -\frac{\partial}{\partial x} \sigma(x, E) p^s + \frac{\partial}{\partial E} \left( \gamma \dot{x}^2 / \sigma(x, E) - \gamma \right) p^s + \alpha \frac{\partial^2}{\partial E^2} \sigma^2(x, E) p^s
\]

The stationary solution:

\[
p^s(x, E) = \frac{N}{|\sigma(x, E)|} \exp \left( -\frac{yE}{\gamma} \right).
\]
Integration over $x$ in regions where $E > U(x)$ removes the dependence of the stationary probability density on the coordinate:

$$p^s(E) \sim T(E) \exp\left(-\frac{\gamma E}{\mathcal{G}}\right)$$

with $T(E)$ being the period of a closed trajectory for a given energy $E$. In particular, for a Hamiltonian function containing only quadratic terms, the stationary probability density is

$$p^s(E) \sim E^{\frac{d}{2} - 1} \exp\left(-\frac{\gamma E}{\mathcal{G}}\right)$$