English Summary

1.1 Dynamical system

vectorfield $\underline{F}: \mathbb{R}^n \times \mathbb{R}_t \to \mathbb{R}^n$; $\underline{x} \in \mathbb{R}^n$ dynamical variable

described as set of (ordinary) differential equations: $\dot{\underline{x}}(t) = \underline{F}(\underline{x}(t), t)$

flow ϕ of vectorfield $\underline{F}$ on manifold M: $\phi: M \times \mathbb{R}_t \to M$ ← think $\mathbb{R}^n$

with $\phi(\underline{x}_0, t) = \phi_t(\underline{x}_0) = \underline{x}(t; \underline{x}_0)$ ← initial condition

fixed point: $\dot{\underline{x}} = 0 \Rightarrow \underline{F}(\underline{x}^*) = 0$

linear stability analysis: $\delta\dot{\underline{x}} = \frac{d}{dt}(\underline{x} - \underline{x}^*) \approx \overbrace{(\underline{\underline{DF}})}^{\text{Jacobian matrix}}\Big|_{\underline{x}^*} \underline{\delta x}$

ansatz: $\underline{\delta x} = \sum_{k=1}^{n} c_k \underline{\xi}^{(k)} e^{\lambda_k t}$ with $\lambda_k \underline{\xi}^{(k)} = \underline{\underline{DF}}\Big|_{\underline{x}^*} \underline{\xi}^{(k)}$ $\quad k = 1, \dots, n$

Eigenvalue (λ_k), eigenvector ($\underline{\xi}^{(k)}$)

example: SIR model with demography (Susceptible, Infected, Recovered)

$$\dot{S} = \mu - \beta S I - \mu S$$
$$\dot{I} = \beta S I - \gamma I - \mu I$$
$$\dot{R} = \gamma I - \mu R$$

β: infection rate
γ: recovery rate
μ: birth/death rate

$S + I + R = 1$

$S, I, R \geq 0$

case 1: $\mu = 0 \Rightarrow I^* = 0 \Rightarrow S, I = \text{constant}$ $\quad S(0) < \frac{\gamma}{\beta} =: \frac{1}{R_0}$ disease dies out (R_0 ← basic reproduction number)

case 2: $\mu > 0$: redefine $R_0 = \frac{\beta}{\gamma + \mu}$ $\quad S(0) > \frac{\gamma}{\beta}$ outbreak

fixed points: (i) $I^* = 0$: disease free

(ii) $S^* = \frac{1}{R_0} = \frac{\gamma + \mu}{\beta} \Rightarrow I^* = \frac{\mu}{\beta}(R_0 - 1) \Rightarrow R^* = \frac{\gamma}{\beta}(R_0 - 1)$ endemic

1.1 Dynamische Systeme (Fortsetzung)

Stabilitätsanalyse: Bestimme Eigenwerte der Jacobi-Matrix

$$\underline{\underline{DF}}\Big|_{\underline{x}^*} = \begin{pmatrix} -\beta I^* - \mu & -\beta S^* & 0 \\ \beta I^* & \beta S^* - (\mu + \gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix}$$

⇒ Lösen der charakteristischen Gleichung: $0 \overset{!}{=} \det\left(\underline{\underline{DF}}\Big|_{\underline{x}^*} - \underbrace{\Lambda}_{\text{Eigenwert}} \underbrace{\mathbb{1}}_{3\times3\text{ Einheitsmatrix}}\right)$

$$0 = (-\beta I^* - \mu - \Lambda)(\beta S^* - \mu - \gamma - \Lambda)(-\mu - \Lambda) - (-\beta S^*)(\beta I^*)(-\mu - \Lambda)$$
$$= \underbrace{(-\mu - \Lambda)}\left[(-\beta I^* - \mu - \Lambda)(\beta S^* - \mu - \gamma - \Lambda) + \beta S^* \beta I^*\right]$$

$\Rightarrow \Lambda_1 = -\mu < 0$ (stabile Richtung)

$\Lambda_{2,3}$ durch Lösen der quadratischen Gleichung $[\ldots]=0$

Fall (i) $I^* = 0,\ S^* = 1$ (Krankheitsfrei)

$$\Rightarrow 0 = \underbrace{(-\mu-\Lambda)}\underbrace{(\beta-\mu-\gamma-\Lambda)}$$

$$\Rightarrow \Lambda_2 = -\mu < 0 \qquad \Rightarrow \Lambda_3 = \beta-\mu-\gamma$$

Fixpunkt ist stabil $\Lambda_3 < 0 \Rightarrow \beta < \mu+\gamma \Leftrightarrow \frac{\beta}{\gamma+\mu} = R_0 < 1$

Fall (ii) $S^* = \frac{1}{R_0},\ I^* = \frac{\mu}{\beta}(R_0-1),\ R^* = \frac{\gamma}{\beta}(R_0-1)$ (endemisch)

$$\Rightarrow 0 = [\ldots] = \Lambda^2 + \mu R_0 \Lambda + \mu(R_0-1)(\gamma+\mu)$$

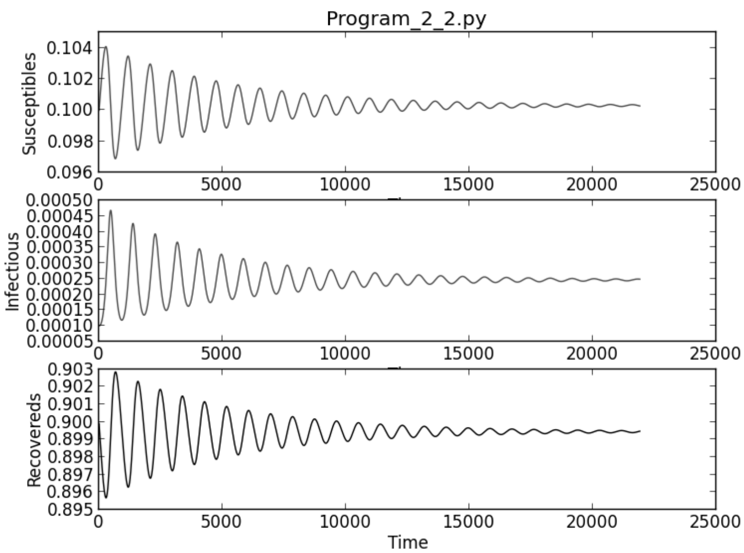
$$\Rightarrow \Lambda_{2,3} = -\frac{\mu R_0}{2} \pm \sqrt{\left(\frac{\mu R_0}{2}\right)^2 - \underbrace{\mu(R_0-1)}_{=:\frac{1}{A}\text{ Durchschnittsalter bei Ansteckung}}\underbrace{(\gamma+\mu)}_{=:\frac{1}{G}}}$$

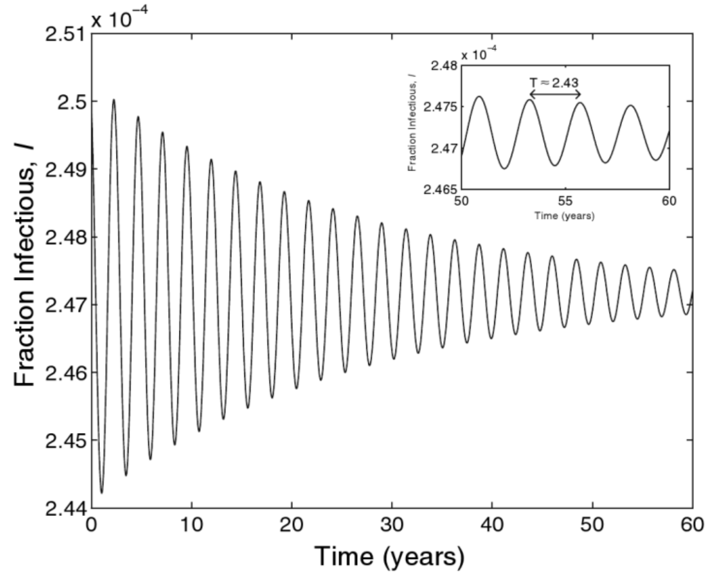
$$\frac{1}{\beta I^*} = \frac{1}{\mu(R_0-1)} = A \Rightarrow \underbrace{R_0 - 1 = \frac{1}{\mu A} = \frac{\text{Lebensdauer}}{A}}_{\text{siehe Tabelle } R_0 \text{ aus Vorlesung 2}}$$

$$\Lambda_1 = -\mu,\quad \Lambda_{2,3} = -\frac{\mu R_0}{2} \pm \frac{\sqrt{(\mu R_0)^2 - 4/AG}}{2} \underset{\mu R_0 \text{ klein}}{\approx} -\frac{\mu R_0}{2} \pm \frac{i}{\sqrt{AG}}$$

$\Rightarrow$ endemische Fixpunkt ist stabil, weil $Re(\Lambda_{2,3}) < 0$

$$Im(\Lambda_{2,3}) \approx \pm\frac{1}{\sqrt{AG}} \Rightarrow T = \frac{2\pi}{Im\Lambda_{2,3}} = \pm 2\pi\sqrt{AG} = \pm\frac{2\pi}{\sqrt{\mu(R_0-1)(\gamma+\mu)}}$$





Parameter: $\frac{1}{\mu} = 70a$, $\beta = 520/a$, $\frac{1}{\gamma} = 7d \Rightarrow R_0 \approx 10$, $I(0) = 2.5 \cdot 10^{-4}$, $S(0) = 0.1$

$\Rightarrow T = 2.43a$

1.2 Einführung zu Netzwerken

Bsp: sozial: Facebook, Freundschaften, www, Koautorenschaft, Zitate...

technische: Strom, Internet, Verkehr, Transport ...

biologische: Gehirn, Klima, Tierhandel ...

⋮

$\Rightarrow$ Netzwerke sind überall

Dynamik von Netzwerken $\Leftrightarrow$ Dynamik auf Netzwerken

	von	auf
Struktur	variable	fest
Knotendynamik	fest	variable
#Knoten/Links	variable	fest
Funktionalität	Netzwerk	Knotenzustand

Kombinierbar durch Wechselwirkungen: Rückkopplung der Dynamik auf Netzwerkstruktur

$\Rightarrow$ adaptive Netzwerke

Einführende Literatur:
- M. E. J. Newman: Networks: An Introduction
- A.-L. Barabasi: Network Science

=> barabasi.com/networksciencebook

=> Vorlesungsfolien: ——— . ——— /slides.zip

Adjazenzmatrix: $\underline{\underline{A}} = \{a_{ij}\}_{i,j=1,\dots,N}$ mit $a_{ij} = \begin{cases} 1 & \text{wenn ein Link von Knoten } j \text{ nach } i \\ 0 & \text{sonst} \end{cases}$

Notation aus linearer Algebra (Vektor-Matrix-Multiplikation): $\frac{d}{dt}\begin{pmatrix} \vdots \\ x_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \\ \text{i-te Z} \\ \\ \end{pmatrix}\begin{pmatrix} \vdots \\ x_j \\ \vdots \end{pmatrix}$

$$\dot{x}_i = F_i(x_i) + \sum_{j=1}^{N} a_{ij} x_j$$

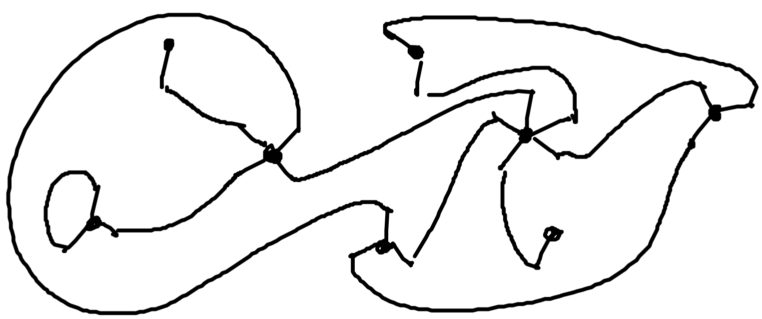
Achtung: Graphentheorie verwendet $a_{ij} = 1$, wenn Link von i nach j existiert (Transformation/Umrechnung durch Transponieren von $\underline{\underline{A}}$)

Eingangsgrad eines Knoten i: $k_i^{(in)} = \sum_{j=1}^{N} a_{ij}$ Zeilensumme

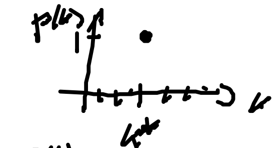
Ausgangsgrad eines Knoten i: $k_i^{(out)} = \sum_{j=1}^{N} a_{ji}$ Spaltensumme

Konstruktion eines Netzwerks mit vorgegebener Gradverteilung (configuration model)

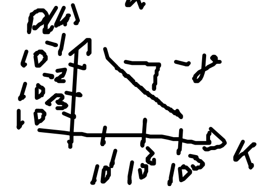
k	1	2	3	4	5
$p(k)$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$



reguläres Netzwerk : $p(k) = \delta_{k,k^*}$



skalenfreies Netzwerk: $p(k) \sim k^{-\gamma}$



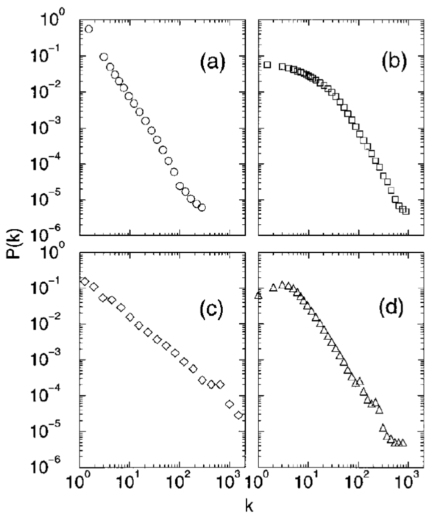


FIG. 3. The degree distribution of several real networks: (a) Internet at the router level. Data courtesy of Ramesh Govindan; (b) movie actor collaboration network. After Barabási and Albert 1999. Note that if TV series are included as well, which aggregate a large number of actors, an exponential cutoff emerges for large k (Amaral *et al.*, 2000); (c) co-authorship network of high-energy physicists. After Newman (2001a, 2001b); (d) co-authorship network of neuroscientists. After Barabási *et al.* (2001).

TABLE II. The scaling exponents characterizing the degree distribution of several scale-free networks, for which $P(k)$ follows a power law (2). We indicate the size of the network, its average degree $\langle k \rangle$, and the cutoff κ for the power-law scaling. For directed networks we list separately the indegree (γ_{in}) and outdegree (γ_{out}) exponents, while for the undirected networks, marked with an asterisk (*), these values are identical. The columns l_{real}, l_{rand}, and l_{pow} compare the average path lengths of real networks with power-law degree distribution and the predictions of random-graph theory (17) and of Newman, Strogatz, and Watts (2001) [also see Eq. (63) above], as discussed in Sec. V. The numbers in the last column are keyed to the symbols in Figs. 8 and 9.

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}	ℓ_{real}	ℓ_{rand}	ℓ_{pow}	Reference	Nr.
WWW	325 729	4.51	900	2.45	2.1	11.2	8.32	4.77	Albert, Jeong, and Barabási 1999	1
WWW	4×10^7	7		2.38	2.1				Kumar *et al.*, 1999	2
WWW	2×10^8	7.5	4000	2.72	2.1	16	8.85	7.61	Broder *et al.*, 2000	3
WWW, site	260 000				1.94				Huberman and Adamic, 2000	4
Internet, domain*	3015–4389	3.42–3.76	30–40	2.1–2.2	2.1–2.2	4	6.3	5.2	Faloutsos, 1999	5
Internet, router*	3888	2.57	30	2.48	2.48	12.15	8.75	7.67	Faloutsos, 1999	6
Internet, router*	150 000	2.66	60	2.4	2.4	11	12.8	7.47	Govindan, 2000	7
Movie actors*	212 250	28.78	900	2.3	2.3	4.54	3.65	4.01	Barabási and Albert, 1999	8
Co-authors, SPIRES*	56 627	173	1100	1.2	1.2	4	2.12	1.95	Newman, 2001b	9
Co-authors, neuro.*	209 293	11.54	400	2.1	2.1	6	5.01	3.86	Barabási *et al.*, 2001	10
Co-authors, math.*	70 975	3.9	120	2.5	2.5	9.5	8.2	6.53	Barabási *et al.*, 2001	11
Sexual contacts*	2810			3.4	3.4				Liljeros *et al.*, 2001	12
Metabolic, *E. coli*	778	7.4	110	2.2	2.2	3.2	3.32	2.89	Jeong *et al.*, 2000	13
Protein, *S. cerev.**	1870	2.39		2.4	2.4				Jeong, Mason, *et al.*, 2001	14
Ythan estuary*	134	8.7	35	1.05	1.05	2.43	2.26	1.71	Montoya and Solé, 2000	14
Silwood Park*	154	4.75	27	1.13	1.13	3.4	3.23	2	Montoya and Solé, 2000	16
Citation	783 339	8.57			3				Redner, 1998	17
Phone call	53×10^6	3.16		2.1	2.1				Aiello *et al.*, 2000	18
Words, co-occurrence*	460 902	70.13		2.7	2.7				Ferrer i Cancho and Solé, 2001	19
Words, synonyms*	22 311	13.48		2.8	2.8				Yook *et al.*, 2001b	20

Exponenten γ liegen häufig im Bereich von 2

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Statistical mechanics of complex networks

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