

English Summary

1.1 Dynamical system

described as set of (ordinary) differential equations: $\dot{\underline{x}}(t) = \underline{F}(\underline{x}(t), t)$

flow ϕ of vectorfield \underline{F} on manifold M : $\phi: M \times \mathbb{R}_t \rightarrow M$ - think \mathbb{R}^4 initial with $\phi(\underline{x}_0, t) = \phi_t(\underline{x}_0) = \underline{x}(t, \underline{x}_0)$ ^{constant}

fixed point: $\dot{\underline{x}} = 0 \Rightarrow \underline{F}(\underline{x}^*) = 0$

linear stability analysis: $\dot{\underline{x}} = \frac{d}{dt}(\underline{x} - \underline{x}^*) \approx \left(\frac{DF}{d\underline{x}} \right)_{\underline{x}^*} \underline{\delta x}$

ansatz: $\underline{\delta x} = \sum_{k=1}^n \underline{\xi}^{(k)} e^{\lambda_k t}$ with $\lambda_k \underline{\xi}^{(k)} = \frac{DF}{d\underline{x}} \bigg|_{\underline{x}^*} \underline{\xi}^{(k)}$ $k=1, \dots, n$
Eigenvalue Eigenvektor

Example: SIR model with demography (Susceptible, Infected, Recovered)

$$\dot{S} = \mu - \beta SI - \mu S$$

$$\dot{I} = \beta SI - \gamma I - \mu I$$

$$\dot{R} = \gamma I - \mu R$$

β : infection rate

γ : recovery rate

μ : birth/death rate

$$S + I + R = 1$$

$$S, I, R \geq 0$$

Case 1: $\mu = 0 \Rightarrow I^* = 0 \Rightarrow S, I = \text{constant}$ $S(t) < \frac{\gamma}{\beta} =: \frac{1}{R_0}$ disease dies out

Case 2: $\mu > 0$: redefine $R_0 = \frac{\beta}{\gamma + \mu}$ $S(t) > \frac{\gamma}{\beta}$ outbreak

fixed points: (i) $I^* = 0$: disease free

(ii) $S^* = \frac{1}{R_0} = \frac{\gamma + \mu}{\beta} \Rightarrow I^* = \frac{\mu}{\beta} (R_0 - 1) \Rightarrow R^* = \frac{\gamma}{\beta} (R_0 - 1)$ endemic

1.1 Dynamische Systeme (Fortsetzung)

Stabilitätsanalyse: Bestimme Eigenwerte der Jacobi-Matrix

$$\frac{DF}{d\underline{x}} \bigg|_{\underline{x}^*} = \begin{pmatrix} -\beta I^* - \mu & -\beta S^* & 0 \\ \beta I^* & \beta S^* - (\mu + \gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix}$$

Eigenwert $\lambda_1 = -\mu$ λ_2, λ_3 Eigenwerte

\Rightarrow Lösen der charakteristischen Gleichung: $0 = \det \left(\frac{DF}{d\underline{x}} \bigg|_{\underline{x}^*} - \lambda \mathbb{1} \right)$

$$0 = (-\beta I^* - \mu - \lambda) (\beta S^* - \mu - \gamma - \lambda) (-\mu - \lambda) - (-\beta S^*) (\beta I^*) (-\mu - \lambda)$$

$$= (-\mu - \lambda) \left[(-\beta I^* - \mu - \lambda) (\beta S^* - \mu - \gamma - \lambda) + \beta S^* \beta I^* \right]$$

$\Rightarrow \lambda_1 = -\mu < 0$ (stabile Richtung)

$\lambda_{2,3}$ durch Lösen der quadratischen Gleichung $[\dots] = 0$

Fall (i) $I^* = 0, S^* = 1$ (krankheitsfrei)

$$\Rightarrow 0 = \underbrace{(-\mu - 1)}_{\lambda_2} \underbrace{(\beta - \mu - \gamma - 1)}_{\lambda_3}$$

$$\Rightarrow \lambda_2 = -\mu < 0 \quad \Rightarrow \lambda_3 = \beta - \mu - \gamma$$

Fixpunkt ist stabil $\lambda_3 < 0 \Rightarrow \beta < \mu + \gamma \Leftrightarrow \frac{\beta}{\gamma + \mu} = R_0 < 1$

Fall (ii) $S^* = \frac{1}{R_0}, I^* = \frac{\mu}{\beta} (R_0 - 1), R^* = \frac{\gamma}{\beta} (R_0 - 1)$ (endemisch)

$$\Rightarrow 0 = [\dots] = \lambda^2 + \mu R_0 \lambda + \mu (R_0 - 1) (\gamma + \mu)$$

$$\Rightarrow \lambda_{2,3} = -\frac{\mu R_0}{2} \pm \sqrt{\left(\frac{\mu R_0}{2}\right)^2 - \mu (R_0 - 1) (\gamma + \mu)} =: \frac{1}{G}$$

$=: \frac{1}{A}$ Durchschnittsalter bei Aussterben

$$\frac{1}{\beta I^*} = \frac{1}{\mu (R_0 - 1)} = A \Rightarrow R_0 - 1 = \frac{1}{\mu A} = \frac{\text{Lebensdauer}}{A}$$

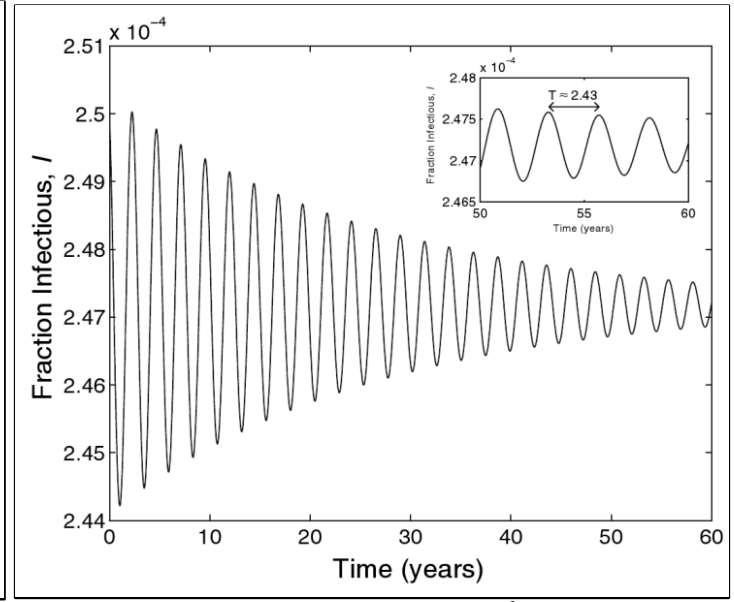
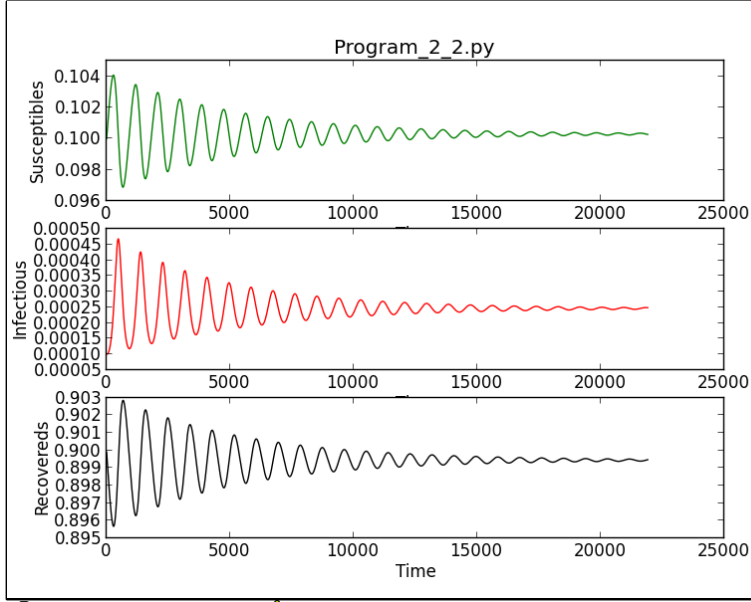
Siehe Tabelle R_0 aus Vorlesung 2

$$\lambda_1 = -\mu, \quad \lambda_{2,3} = -\frac{\mu R_0}{2} \pm \frac{\sqrt{(\mu R_0)^2 - 4/A G}}{2} \approx -\frac{\mu R_0}{2} \pm \frac{i}{\sqrt{A G}}$$

$\mu R_0 \ll 1$

\Rightarrow endemische Fixpunkt ist stabil, weil $\text{Re}(\lambda_{2,3}) < 0$

$$\text{Im}(\lambda_{2,3}) \approx \pm \frac{1}{\sqrt{A G}} \Rightarrow T = \frac{2\pi}{\text{Im} \lambda_{2,3}} = \pm 2\pi \sqrt{A G} = \pm \frac{2\pi}{\sqrt{\mu (R_0 - 1) (\gamma + \mu)}}$$



Parameter: $\mu = 70/a$, $\beta = 520/a$, $\frac{1}{\gamma} = 7d \Rightarrow R_0 = 10$, $I(0) = 2.5 \cdot 10^{-4}$, $S(0) = 0.1$
 $\Rightarrow T = 2.43a$

1.2 Einführung zu Netzwerken

- Bsp; sozial: Facebook, Freundschaften, WWW, Kooperationschaft, Zitate...
- technisch: Strom, Internet, Verkehr, Transport...
- biologisches Gehirn, Klima, Tierhandel...
- ⋮

\Rightarrow Netzwerke sind überall

Dynamik von Netzwerken \leftrightarrow Dynamik auf Netzwerken

	von	auf
Struktur	variable	fest
Knotendynamik	fest	variable
# Knoten/Linien	variable	fest
Funktionalität Netzwerk		Knotenzustand

Kombinierbar durch Wechselwirkungen: Rückkopplung der Dynamik auf Netzwerkstruktur \Rightarrow adaptive Netzwerke

Einführende Literatur: M. E. J. Newman: Networks: An Introduction

• A.-L. Barabasi: Network Science

⇒ barabasi.com/networksciencebook

⇒ Vorlesungsfolien: — v — slides ⇒ ip

Adjazenzmatrix: $\underline{A} = \{a_{ij}\}_{i,j=1,\dots,N}$ mit $a_{ij} = \begin{cases} 1 & \text{wenn ein Link von Knoten } j \text{ nach } i \\ 0 & \text{sonst} \end{cases}$

Notation aus linearer Algebra (Vektor-Matrix-Multiplikation): $\frac{d}{dt} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} -k_1 x_1 \\ \vdots \\ -k_N x_N \end{pmatrix}$

$$\dot{x}_i = F_i(x_i) + \sum_{j=1}^N a_{ij} x_j$$

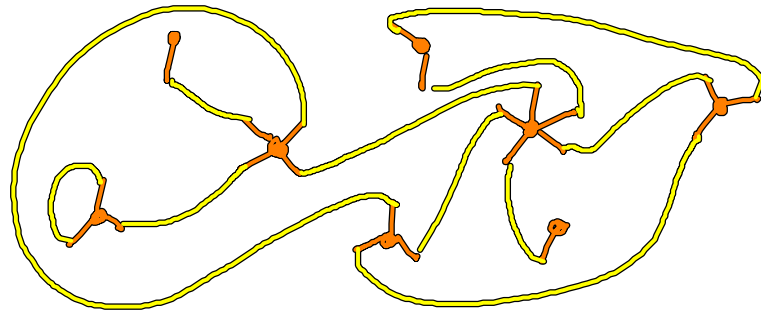
Achtung: Graphentheorie verwendet $a_{ij} = 1$, wenn Link von i nach j existiert
(Transformation/Umkehrung durch Transponieren von \underline{A})

Eingangsgrad eines Knoten i : $k_i^{(in)} = \sum_{j=1}^N a_{ij}$ Zeilensumme

Ausgangsgrad eines Knoten i : $k_i^{(out)} = \sum_{j=1}^N a_{ji}$ Spaltensumme

Konstruktion eines Networks mit vorgegebener Gradbarkeit (configuration model)

k	1	2	3	4	5
n_k	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$



reguläres Netzwerk: $p(k) = \frac{1}{N} \sum_k k^{\alpha}$

skalenfreies Netzwerk: $p(k) \sim k^{-\beta}$

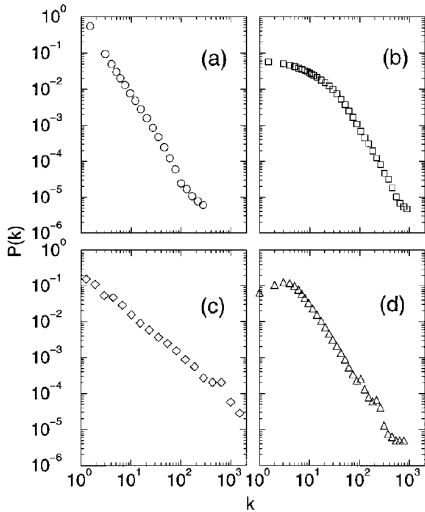
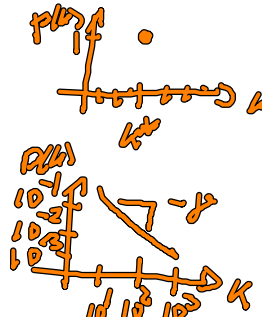


FIG. 3. The degree distribution of several real networks: (a) Internet at the router level. Data courtesy of Ramesh Govindan; (b) movie actor collaboration network. After Barabási and Albert 1999. Note that if TV series are included as well, which aggregate a large number of actors, an exponential cut-off emerges for large k (Amaral *et al.*, 2000); (c) co-authorship network of high-energy physicists. After Newman (2001a, 2001b); (d) co-authorship network of neuroscientists. After Barabási *et al.* (2001).

TABLE II. The scaling exponents characterizing the degree distribution of several scale-free networks, for which $P(k)$ follows a power law (2). We indicate the size of the network, its average degree ($\langle k \rangle$), and the cutoff κ for the power-law scaling. For directed networks we list separately the indegree (γ_{in}) and outdegree (γ_{out}) exponents, while for the undirected networks, marked with an asterisk (*), these values are identical. The columns ℓ_{real} , ℓ_{rand} , and ℓ_{pow} compare the average path lengths of real networks with power-law degree distribution and the predictions of random-graph theory (17) and of Newman, Strogatz, and Watts (2001) [also see Eq. (63) above], as discussed in Sec. V. The numbers in the last column are keyed to the symbols in Figs. 8 and 9.

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}	ℓ_{real}	ℓ_{rand}	ℓ_{pow}	Reference	Nr.
WWW	325 729	4.51	900	2.45	2.1	11.2	8.32	4.77	Albert, Jeong, and Barabási 1999	1
WWW	4×10^7	7		2.38	2.1				Kumar <i>et al.</i> , 1999	2
WWW	2×10^8	7.5	4000	2.72	2.1	16	8.85	7.61	Broder <i>et al.</i> , 2000	3
WWW, site	260 000				1.94				Huberman and Adamic, 2000	4
Internet, domain*	3015–4389	3.42–3.76	30–40	2.1–2.2	2.1–2.2	4	6.3	5.2	Faloutsos, 1999	5
Internet, router*	3888	2.57	30	2.48	2.48	12.15	8.75	7.67	Faloutsos, 1999	6
Internet, router*	150 000	2.66	60	2.4	2.4	11	12.8	7.47	Govindan, 2000	7
Movie actors*	212 250	28.78	900	2.3	2.3	4.54	3.65	4.01	Barabási and Albert, 1999	8
Co-authors, SPIRES*	56 627	173	1100	1.2	1.2	4	2.12	1.95	Newman, 2001b	9
Co-authors, neuro.*	209 293	11.54	400	2.1	2.1	6	5.01	3.86	Barabási <i>et al.</i> , 2001	10
Co-authors, math.*	70 975	3.9	120	2.5	2.5	9.5	8.2	6.53	Barabási <i>et al.</i> , 2001	11
Sexual contacts*	2810			3.4	3.4				Liljeros <i>et al.</i> , 2001	12
Metabolic, <i>E. coli</i>	778	7.4	110	2.2	2.2	3.2	3.32	2.89	Jeong <i>et al.</i> , 2000	13
Protein, <i>S. cerev.</i> *	1870	2.39		2.4	2.4				Jeong, Mason, <i>et al.</i> , 2001	14
Ythan estuary*	134	8.7	35	1.05	1.05	2.43	2.26	1.71	Montoya and Solé, 2000	14
Silwood Park*	154	4.75	27	1.13	1.13	3.4	3.23	2	Montoya and Solé, 2000	16
Citation	783 339	8.57			3				Redner, 1998	17
Phone call	53×10^6	3.16		2.1	2.1				Aiello <i>et al.</i> , 2000	18
Words, co-occurrence*	460 902	70.13		2.7	2.7				Ferrer i Cancho and Solé, 2001	19
Words, synonyms*	22 311	13.48		2.8	2.8				Yook <i>et al.</i> , 2001b	20

Exponenten & Längen häufig in Besatz von 2

Statistical mechanics of complex networks

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