

## Transport through coupled quantum dots

**Tobias Brandes**

Institut für Theoretische Physik, Technische Universität Berlin, Hardenbergstr. 36, 10623 Berlin, Germany

Received 20 February 2006, revised 20 April 2006, accepted 20 April 2006

Published online 12 July 2006

**PACS** 72.70.+m, 73.21.La, 73.23.Hk

A brief survey of theoretical activities on electronic transport in coupled charge qubits is given. Some subtleties of the double quantum dot model (infinite bias, infinitely strong Coulomb blockade) are pointed out, and the motivation behind using ac-field, noise and entanglement as a probe for testing transport in a simple setting is discussed.

phys. stat. sol. (b) **243**, No. 10, 2293–2296 (2006) / DOI 10.1002/pssb.200668050

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### 1 Introduction

In a somewhat ironical twist of history, the discovery of the quantum Hall effects in ‘dirty’ condensed matter systems has changed the perception of electronic transport into something where ‘pure’ and (sometimes) very elegant quantum physics can be displayed within the most complicated environments. Quantum transport theory, of course, notoriously difficult and plagued by the complicated interplay between interactions among non-equilibrium electrons. One may perhaps savely say that no-one has ever reliably calculated a quantity like, say, the conductivity of a metal: many-body systems simply are no easy fellows.

Fortunately enough, the recent trend towards realising quantum superpositions and entanglement offers some new playground, also (or perhaps even mainly?) for theorists, to ‘re-discover’ low-dimensional models – models where electronic degrees of freedom are frozen out down to as little as two, just enough to do quantum physics. Still, coupling to bosonic environments,  $1/f$  noise, radiation fields, and external leads leaves room for sufficiently complicated and interesting physics. In the following, parts of this physics will be discussed for the particular example of semiconductor charge qubits in the strong Coulomb blockade regime [1].

The arguably simplest quantum transport case occurs for spin-polarised electrons in two tunnel-coupled levels  $|L\rangle$  (left) and  $|R\rangle$  (right) attached to fermionic reservoirs. With the ‘empty’ state  $|0\rangle$  and the transport model

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_{\text{res}} + \mathcal{H}_T, \quad (1)$$

$$\mathcal{H}_S = \frac{\varepsilon}{2} \hat{\sigma}_z + T_c \hat{\sigma}_x, \quad \hat{\sigma}_z = |L\rangle \langle L| - |R\rangle \langle R|, \quad \hat{\sigma}_x = |L\rangle \langle R| + |R\rangle \langle L|, \quad (2)$$

$$\mathcal{H}_{\text{res}} = \sum_{k,j=L,R} \varepsilon_k c_{k,i}^\dagger c_{k,i}, \quad \mathcal{H}_T = \sum_{k,j=L,R} (V_k^i c_{k,i}^\dagger |0\rangle \langle i| + \text{H.c.}), \quad (3)$$

\* e-mail: brandes@physik.tu-berlin.de

one immediately faces a quantum impurity problem due to the orbital degree of freedom ('pseudo-spin', L-R) that eventually runs into Kondo-physics at low-temperatures (including spin, this has been discussed recently as an 'SU(4) Kondo model' [2]). Gurvitz [3, 4] and Stoof and Nazarov [5] introduced the limit of infinite source–drain bias for this model which then becomes exactly solvable in terms of a master equation: co-tunneling processes can then be safely neglected—electrons are injected from the left one-by-one and tunnel from the right dot into a Fermi vacuum. Mathematically, the various limits invoked here look quite frightening: (1) magnetic field  $B \rightarrow \infty$  in order to spin-polarize the electrons; (2) Coulomb interaction  $U \rightarrow \infty$  in order to just allow one additional 'transported' electron into the dots; (3) chemical potential difference (left-right reservoir bias)  $\mu_L - \mu_R \rightarrow \infty$  in order to suppress Kondo-physics; (4) band-width over which the tunneling density of states is constant  $\delta E \rightarrow \infty$  in order to invoke the Markov approximation in the resulting generalised Master equation. Only the limit (4) is not strictly necessary in order to achieve an exact solution, but the question of whether or not a non-Markovian description is valid here seems to not yet have been discussed in the literature.

In spite of these uncertainties, the model Eq. (1) and its extensions have been used by various groups, mainly owing to its simplicity, as a test-bed for various many-body methods [6, 7], as a tool to extend quantum optics concepts such as adiabatic steering, Rabi-oscillations etc. into the electron transport realm [8], but also as a result of a series of beautiful experiments on double quantum dots to which at least some of the theoretical predictions actually appear to be applicable [1]. The first of these extensions comprises an addition of deterministic (classical) as well as quantum fluctuations of the energy level difference  $\varepsilon$  in Eq. (1),

$$\varepsilon \rightarrow \varepsilon + \Delta \cos \Omega t + \sum_{\mathbf{Q}} g_{\mathbf{Q}} (a_{-\mathbf{Q}} + a_{\mathbf{Q}}^{\dagger}). \quad (4)$$

Here,  $\Delta$  is the strength of a (microwave frequency) electric field of angular frequency  $\Omega$ , whereas  $g_{\mathbf{Q}}$  is an electron–phonon coupling matrix element and  $a_{\mathbf{Q}}^{\dagger}$  creates a phonon of wave vector  $\mathbf{Q}$ . The resulting time-dependent Hamiltonian then is a transport version of a driven spin-boson problem [9]. More generally, one can write the (double dot) system Hamiltonian in a pseudo-spin form

$$\mathcal{H}_S(t) = -\hat{\mu} \mathbf{B}(t), \quad (5)$$

where  $\hat{\mu}$  is the vector of the Pauli spin matrices in the L-R basis and  $\mathbf{B}(t)$  contains all the parameters of the system, including fluctuating bosonic degrees of freedom. Of course, even the isolated system  $\mathcal{H}_S(t)$  without coupling to fermionic reservoirs is in general a complicated problem, not to speak of its transport version.

## 2 Bloch-sphere representation, Zeno effect

The solution of the Master equation for the stationary state  $\rho(t \rightarrow \infty)$  of the reduced double-dot density operator is

$$\rho_{RR} = T_c^2 / \mathcal{N}, \quad \rho_{LL} = (T_c^2 + \varepsilon^2 + \Gamma_R^2 / 4) / \mathcal{N}, \quad \rho_{LR} = T_c (\varepsilon - i\Gamma_R / 2) / \mathcal{N}, \quad (6)$$

$$\mathcal{N} = \varepsilon^2 + \Gamma_R^2 / 4 + T_c^2 (2 + \Gamma_R / \Gamma_L), \quad (7)$$

which for example yields the stationary current  $I = 2eT_c \text{Im} \rho_{LR}$ , where  $-e < 0$  is the electron charge and  $\Gamma_i = 2\pi \sum_k |V_k^i| \delta(\varepsilon - \varepsilon_k)$  ( $i = L, R$ ) are the tunnel rates. A projection  $\hat{P}\rho(t \rightarrow \infty)$  is now defined on states with exactly one additional electron, i.e. projecting out the empty subspace  $|0\rangle\langle 0|$  which does not couple to the other states in the Master equation (this decoupling is an example of a charge superselection rule 'in action'). In pseudo-spin language, the properly normalised state  $\hat{P}\rho(t \rightarrow \infty)$  is represented by the Bloch vector

$$\langle \sigma \rangle \equiv (\langle \sigma_x \rangle, \langle \sigma_y \rangle, \langle \sigma_z \rangle) = \left( \frac{2T_c \varepsilon}{\mathcal{N}_{\infty}}, \frac{\Gamma_R T_c}{\mathcal{N}_{\infty}}, \frac{\Gamma_R^2 / 4 + \varepsilon^2}{\mathcal{N}_{\infty}} \right), \quad (8)$$

with  $\mathcal{N}_\infty \equiv \Gamma_R^2/4 + \varepsilon^2 + 2T_c^2$ , which turns out to be just the limit of  $\Gamma_L \rightarrow \infty$  of the full solution at finite  $\Gamma_L$  that has finite occupation  $\rho_{00} = 1 - \rho_{RR} - \rho_{LL}$  of the empty state:  $\hat{P}\rho(t \rightarrow \infty) = \lim_{\Gamma_L \rightarrow \infty} \rho(t \rightarrow \infty)$ .

Some interesting observations can now be made already here: (1) For vanishing internal bias  $\varepsilon = 0$ , the qubit is in a strongly mixed state for *weak* tunnel coupling  $\Gamma_R$  to the right, i.e.  $\langle \sigma \rangle \rightarrow 0$ . (2) For arbitrary internal bias  $\varepsilon$  and *strong* tunnel coupling  $\Gamma_R$  to the right, the system becomes localised in the (pure) left state, i.e.  $\langle \sigma \rangle \rightarrow (0, 0, 1)$ . This is a ‘transport example’ of the *quantum Zeno effect*: the right (empty) lead acts as a broadband detector [10] that continuously measures whether or not an electron is in the right dot [11].

### 3 AC-transport, noise and entanglement

The stationary state  $\rho(t \rightarrow \infty)$  offers not too much insight into the microscopic details of the underlying Hamiltonian  $\mathcal{H}_S$ : for example, it contains no direct information on the level spectrum of  $\mathcal{H}_S$ . An additional, time-dependent field  $\Delta \cos \Omega t$ , cf. Eq. (4), probes the level structure and such a ‘driven’ situation and therefore reveals additional information. A Floquet analysis of the corresponding Master equation in presence of dissipation and driving fields has been carried out in [12]. One central aspect there is the dressing of the tunnel matrix element

$$T_c^2 \rightarrow T_c^2 \sum_{mm'} i^{n'-n} J_n \left( \frac{\Delta}{\Omega} \right) J_{n'} \left( \frac{\Delta}{\Omega} \right) e^{-in\Omega t'} e^{-i(n-n')\Omega t}, \quad (9)$$

leading to coherent destruction of tunneling and an intriguing interplay between photons, phonons, and coherent tunneling. One important result in [12] was the lifting of the usual (perturbative in the tunnel coupling  $T_c^2$ ) Tien–Gordon dynamical localisation by quantum coherence due to higher order terms in  $T_c^2$ , a result consistent with the analysis by Barata and Wreszinski [13] and Frasca [14] for the isolated, driven two-level system. This lifting, however, is a quantum coherent effect and disappears again for increasing dissipation.

A further and possibly even more powerful probe into details of a given quantum system via electronic transport is offered by quantum noise, i.e. the frequency dependent spectrum of current fluctuations [15],

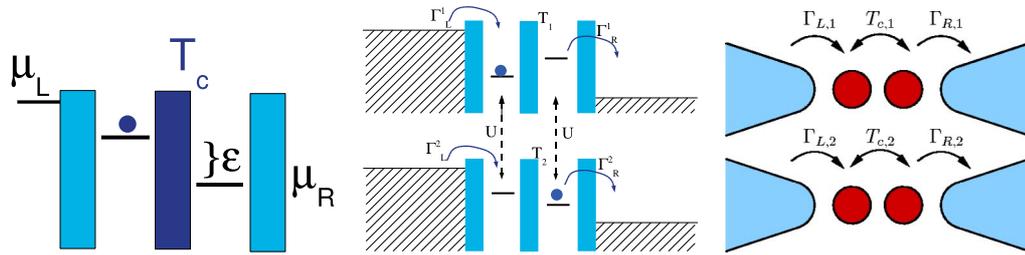
$$S_I(\omega) \equiv \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle \{ \Delta \hat{I}(\tau), \Delta \hat{I}(0) \} \rangle, \quad (10)$$

and the so-called full counting statistics [16, 17], the latter originally stemming from calculations of the emission statistics of photons in quantum optics such as done by Cook for resonance fluorescence [18] in 1981. The pertinent  $n$ -resolved master equations count the number of emission (or tunneling for electrons) events and typically have the form

$$\dot{\rho}_i^{(n)} = i[\mathcal{H}_S, \rho_i^{(n)}] - (L_+ L_- \rho_i^{(n)} + \rho_i^{(n)} L_+ L_- - 2L_- \rho_i^{(n-1)} L_+) \quad (11)$$

with Lindblad operators  $L_\pm$  which are solved via generating operators  $G(s, t) \equiv \sum_{n=0}^{\infty} s^n \rho_i^{(n)}$  with the *counting variable*  $s \equiv e^{i\chi}$ . The counting statistics  $p_n(0, t) = \text{Tr} \rho_i^{(n)}$  is obtained for all times either numerically or via Laplace transformation methods, alternatively one uses approximations for large times  $t \rightarrow \infty$ . The full information content of both  $S_I(\omega)$  and  $p_n(0, t)$  for systems like double quantum dots is currently the subject of very active research, both theoretically and experimentally.

Finally, we conclude this overview by a short discussion of *entanglement* of  $N$  coupled charge qubits, here in particular under (stationary) transport conditions and thereby different from the usual playground of quantum information science. The simplest case of  $N = 2$  is represented in Fig. 1 where entanglement is achieved by direct Coulomb interaction or indirect coupling via the phonon field. The latter case [19] gives rise to an interesting analogy with Dicke sub- and superradiance in trapped ions: already in the simplest cases, the resulting Master equations yield an extremely complex variety of phenomena such as



**Fig. 1** (online colour at: [www.pss-b.com](http://www.pss-b.com)) Left: Double quantum dot model. Center: Two Coulomb-coupled double quantum dots. Right: Two phonon-coupled double quantum dots.

switching on and off of additional resonances that can be analysed in terms of Dicke states and cross-coherences between the double dots. The other case of direct Coulomb coupling via a simple ‘on-site’ interaction offers an opportunity to connect entanglement (via the concurrence of the mixed qubit state projected onto the 2-electron subspace, similar to the discussion in Section 2) and the non-diagonal quantum shot-noise  $S_{ij}(0)$  between the two qubits [20]. It turns out that this ‘non-equilibrium’ entanglement vanishes below  $U \sim 2T_c/\Gamma_R$  where the state is too strongly mixed as to be entangled [21].

Summarising, the ‘philosophy’ behind studying simplified few-level transport models such as Eq. (1) might be described as the hope to better understand some parts of quantum transport – in particular the intriguing interplay between quantum coherence (the internal tunneling  $T_c$ ), non-equilibrium (external fields and reservoirs), dissipation (phonon coupling), and interactions, and the relation of all of these to such fundamental questions as counting and entangling of electrons.

**Acknowledgements** This work is the result of collaborations and many discussions with R. Aguado, Y.-N. Chen, S. Debald, B. Kramer, N. Lambert, G. Platero, and T. Vorrath. A Royal Society Travel Grant (2005) is kindly acknowledged.

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