

Non-equilibrium entanglement in a driven many-body spin-boson model

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Abstract. We study the entanglement dynamics in the externally-driven single-mode Dicke model in the thermodynamic limit, when the field is in resonance with the atoms. We compute the correlations in the atoms-field ground state by means of the density operator that represents the pure state of the universe and the reduced density operator for the atoms, which results from taking the partial trace over the field coordinates. As a measure of bipartite entanglement, we calculate the linear entropy, from which we analyze the entanglement dynamics. In particular, we found a strong relation between the stability of the dynamical parameters and the reported entanglement.

1. Introduction

Understanding the properties displayed by the entanglement of physical systems is one of the fundamental purposes of quantum information theory [1]. A quantum system composed of two or more entangled subsystems has the interesting property that although the state of the total system can be well defined, it is impossible to identify individual properties for each one of its parts. A subject that has attracted recent interest is linked to the relationship between entanglement and certain physical properties of many body quantum systems; this has taken special interest in relation to quantum phase transitions (QPTs) [2, 3, 4]. A QPT in a many body system strongly influences the behavior of the system near to the critical point, with the consequent appearance of long-range correlations in the ground state. Although several proposals exist, currently there is no complete theory of multipartite entanglement, and the common techniques are based on bipartite decompositions of the total system. This type of decomposition has made it possible to study the entanglement in the Dicke model in thermal equilibrium [5, 6, 7]. Here, we study the entanglement properties of a many-body system in the non-equilibrium regime, in which the total system has a unitary dynamics but is not an isolated system. For this purpose we study the single mode “externally-driven” Dicke Hamiltonian

$$\hat{H}(t) = \omega_0 J_z + \omega a^\dagger a + \frac{g(t)}{\sqrt{N}}(J_+ + J_-)(a^\dagger + a), \quad (1)$$

where $J_z = \sum_{i=1}^N J_z^{(i)}$, $J_\pm = \sum_{i=1}^N J_\pm^{(i)}$ are collective atomic operators, ω_0 is the level splitting of the atoms, ω is the frequency of the bosonic mode and $g(t) = g + \Delta g \cos \Omega t$ is the time dependent atom-field coupling. We assume that Δg is a fraction of the static coupling g . An exact

diagonalisation of the problem Eq. (1) has been previously carried out by means of the Holstein-Primakoff transformation for the case of a *static* atom-field coupling $g(t) \equiv g = \text{constant}$ [5, 6], but there is no known solution for the time dependent atom-field coupling case.

2. The reduced density operator

When studying a composed quantum system whose dynamic is unitary, it is interesting to study the dynamics of the parts of the system. In contrast to the dynamics of the whole system, the dynamics of the subsystems is not unitary. However, it is possible to make a description of the subsystem through the reduced density operator. The Schrödinger equation for the single mode “externally-driven” Dicke Hamiltonian has an exact solution when the field is in resonance with the atoms $\omega = \omega_0$. With the purpose of formulating the exact solutions, we introduce an abstract coordinate representation through the coordinates of the physical field x and the atoms coordinate y . In terms of these coordinates, the ground state of the universe (atoms+field) is given by the expression [8]

$$\Psi_{0-,0+}(x, y, t) = \exp(i\gamma_{0-}(t)) \exp(i\gamma_{0+}(t)) \Phi_{0-}^- \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}, t \right) \Phi_{0+}^+ \left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}, t \right), \quad (2)$$

where

$$\Phi_{0\mp}^\mp(w, t) = \left(\frac{1}{2\pi|B^\mp(t)|^2} \right)^{1/4} \exp \left(i \frac{\dot{B}^\mp(t)}{2B^\mp(t)} w^2 \right),$$

and the time dependent phase is given by

$$\gamma_{0\mp}(t) = \int_0^t \langle \Phi_{0\mp}^\mp, t | i \frac{\partial}{\partial t} - \hat{H}(t) | \Phi_{0\mp}^\mp, t \rangle dt.$$

It is interesting to note that the dynamics of the system is influenced by the dynamics of the auxiliary parameters $B^\mp(t)$, which are solutions of the Mathieu equation [9]

$$\ddot{B}^\mp(t) + [(1 \mp 2g) \mp (2\Delta g) \cos \Omega t] B^\mp(t) = 0, \quad (3)$$

subject to the Wronskian condition

$$\dot{B}^\mp(t)(B^\mp)^*(t) - B^\mp(t)(\dot{B}^\mp)^*(t) = i.$$

We describe the dynamics of the total system in terms of the density operator, where the pure state of the universe is represented by the operator $\hat{\rho}_G(t) = |\Psi_{0-,0+}, t\rangle \langle \Psi_{0-,0+}, t|$. In the coordinate representation, the density matrix takes the form

$$\rho_G(x', y'; x, y, t) = \Psi_{0-,0+}^*(x, y, t) \Psi_{0-,0+}(x', y', t). \quad (4)$$

At this stage we develop a bipartite decomposition of the universe, which lets us study the reduced dynamics of the atoms through the reduced density matrix (RDM). In so doing, we calculate the partial trace over the physical field coordinate x ,

$$\rho_G^{(R)}(y', y, t) = \int_{-\infty}^{+\infty} \Psi_{0-,0+}^*(x, y, t) \Psi_{0-,0+}(x, y', t) dx, \quad (5)$$

and as a result of this, the ground state of the universe $\Psi_{0-,0+}(x, y, t)$ given in Eq. (2) is Gaussian, which facilitates the integration of Eq. (5). After some algebraic calculations, we

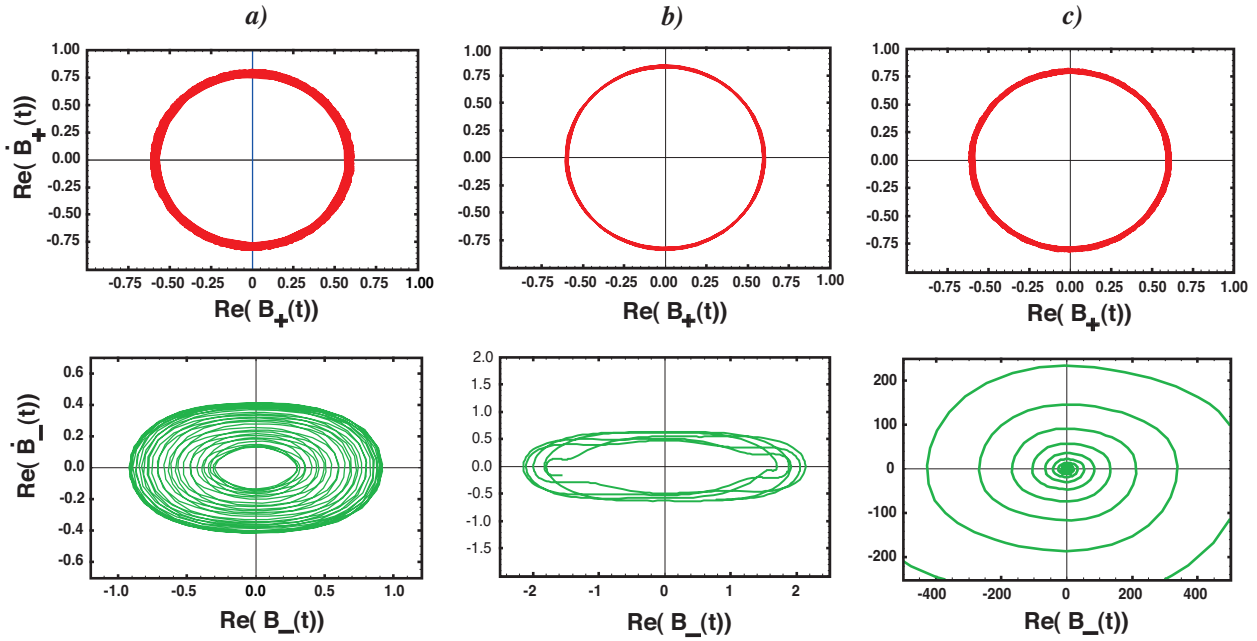


Figure 1. Phase space trajectories of the auxiliary dynamical parameters $B^\pm(t)$ in the case $\Delta g = 0.1g$, for a value of the static coupling in a) the stable region $g = 0.40$, b) the stable region $g = 0.46$, and c) the unstable region $g = 0.38$.

obtain the following result for the reduced density matrix

$$\rho_G^{(R)}(y', y, t) = \left(\frac{\Re(\xi^-)\Re(\xi^+)}{\pi(\Re(\xi^-)c^2 + \Re(\xi^+)s^2)} \right)^{1/2} \exp\left(\frac{c^2s^2[y(\xi^- - \xi^+)^* + y'(\xi^- - \xi^+)]^2}{4(\Re(\xi^-)c^2 + \Re(\xi^+)s^2)} \right) \times \exp\left(-\frac{y^2}{2}[(\xi^-)^*s^2 + (\xi^+)c^2] \right) \exp\left(-\frac{(y')^2}{2}[(\xi^-)s^2 + (\xi^+)c^2] \right), \quad (6)$$

where $c = s = 1/\sqrt{2}$ y $\Re(\xi^\mp)$ is the real part of the function $\xi^\mp(t) = -i\frac{\dot{B}^\mp(t)}{B^\mp(t)}$; in particular, when $\Delta g = 0$ and $\omega = \omega_0 = 1$, this function is independent of time and is defined as $\xi^\mp(t) = \epsilon^\mp = \sqrt{(1 \mp 2g)}$. Hence, our model reproduces the results previously in the literature for the atom's reduced density matrix in the single mode Dicke model in thermal equilibrium [5, 6].

3. The linear entropy

We consider a pure state of a composed system AB (universe) represented by the density operator $\hat{\rho}_{AB}$. The linear entropy for the reduced density operator $\hat{\rho}_A = \text{tr}_B(\hat{\rho}_{AB})$ of the subsystem A is defined as $L_A = 1 - \text{tr}_A(\hat{\rho}_A^2)$. This gives a measure of purity of the reduced density operator $\hat{\rho}_A$ (o $\hat{\rho}_B$) of one part of the total system (in a bipartite decomposition of the universe); therefore, if the pure state of the universe is separable, the reduced density operator of one part of the system represents a pure state and as a result the linear entropy must be zero. Similarly, if the pure state of the universe is a maximally entangled state [1], the linear entropy is 1/2. In the context of the “externally driven” single mode Dicke model, it is possible to use the linear entropy as a measure of bipartite entanglement between the atoms and the field. The linear entropy for the time dependent reduced density operator of the atoms, Eq. (6), is thus given by

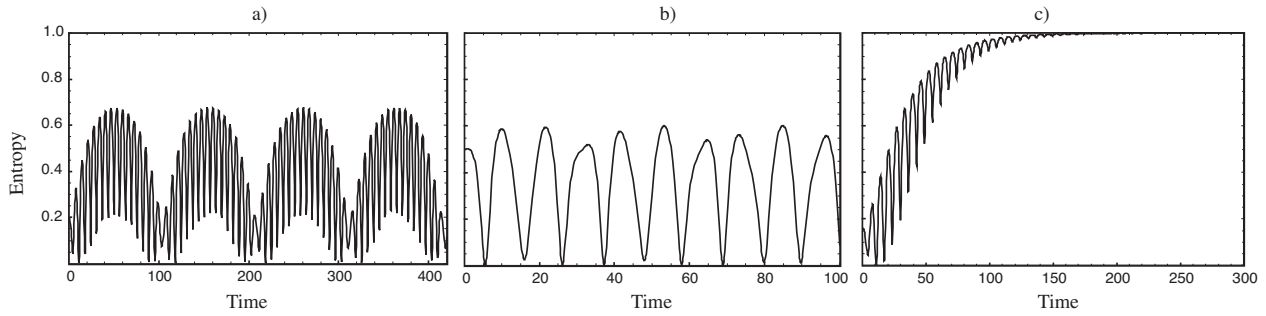


Figure 2. Linear entropy for $\Delta g = 0.1g$; a) $g = 0.40$, b) $g = 0.46$, and c) $g = 0.38$.

$L(t) = 1 - \text{tr} \left[\left(\hat{\rho}_G^{(R)}(t) \right)^2 \right]$. With the aim of calculating the linear entropy explicitly, we can use the representation of the density matrix, Ec. (6)

$$L(t) = 1 - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho_G^{(R)}(y', y, t) \rho_G^{(R)}(y, y', t) dy dy',$$

and after some algebra, we obtain the expression for the linear entropy

$$L(t, g) = 1 - \frac{\pi \Lambda^2}{\sqrt{4(\Re e(\alpha))^2 - \beta^2}}, \quad (7)$$

where the parameters α , β and Λ are defined as

$$\alpha = \frac{(\xi^-)^* s^2 + (\xi^+)^* c^2}{2} - \frac{c^2 s^2 [(\xi^- - \xi^+)^*]^2}{4(\Re e(\xi^-) c^2 + \Re e(\xi^+) s^2)}, \quad \beta = \frac{c^2 s^2 (\xi^- - \xi^+)^* (\xi^- - \xi^+)}{2(\Re e(\xi^-) c^2 + \Re e(\xi^+) s^2)},$$

$$\Lambda = \left(\frac{\Re e(\xi^-) \Re e(\xi^+)}{\pi (\Re e(\xi^-) c^2 + \Re e(\xi^+) s^2)} \right)^{1/2}. \quad (8)$$

4. Results

Interestingly, the linear entropy exhibits a time dependence that is not determined by the global phase of the ground state wave function of the universe, Eq. (2), but by the auxiliary dynamical parameters $B^\mp(t)$. In the particular case $\Delta g = 0.1g$ and $\Omega = 1$, the stability zones of Eq. (3) are known [8], and the study of the corresponding solutions is based on the Floquet's theorem for second order differential equations with time periodic coefficients. For this, the solutions of Eq. (3) have the general form $B^\pm(t) = \exp(iF^\pm t) \phi^\pm(t)$, where $\phi^\pm(t+T) = \phi^\pm(t)$ and F^\pm is the Floquet exponent, which depends on the shape of the driving. For driving functions for which F^\pm is complex, the solution becomes unstable. In the stable regime, F^\pm is a real number [9]. In Figs. 1(a), (b) we consider the phase space representation of the trajectories of the auxiliary dynamical parameters $B^\pm(t)$. For values of g that belong to a common stability zone ($g = 0.4$ and $g = 0.46$) [8] the solutions are bounded in the phase space; in order to establish a relationship between the stability and the entanglement dynamics, we consider the linear entropy in Figs. 2(a), and (b) for the values of the static coupling described above. Our choice of the initial conditions for Eq. (3) implies that the system starts in thermal equilibrium, but is described by a non-separable quantum state. Strikingly, the dynamics exhibits a behaviour

whereby the system experiences a disentanglement process, before reaching a maximum value of entanglement, and then successive collapses and revivals.

Figure 1(c) shows another interesting behaviour: when the static coupling belongs to the instability zone ($g = 0.38$) of $B^-(t)$, the solution $B^+(t)$ is bounded while the solution B^- is unbounded in the phase space. For this value of the static coupling, the linear entropy oscillates, before it reaches the stationary state with linear entropy $L = 1$, as shown in Fig. 2(c). In [8] we discuss the relationship between the stability zones and the localization of the ground state wave function of the universe. In order to study this relation, we define the characteristic length $l^\mp(t) = \sqrt{2}|B^\mp(t)|$, which is bounded in the stable zones and unbounded in the unstable zones. The numerical simulations of the ground state probability density show that for values of the static coupling that belong to the common stability zones the probability is localized in the abstract $x - y$ space and presents an oscillatory behaviour. In contrast, for values of the static coupling that belong to the unstable zones, the probability density has an oscillatory behavior. However, when the system evolves, the density is systematically stretched in a fixed direction, with a consequent dilation in the perpendicular direction.

5. Conclusions

We have obtained exact results for the atoms reduced density operator and the linear entropy for the “externally driven” Dicke model, in the thermodynamic limit. These results allows us to study the entanglement dynamics through the auxiliary dynamical parameters. In contrast, in the Dicke model in thermal equilibrium [5, 6], the atoms-field entanglement does not have dynamics because the temporal dependence of the ground state of the universe is given by a global phase of the wave function. In this work the auxiliary dynamical parameters and its stability properties determine the entanglement dynamics of the system and the behavior of the universe ground state wave packet.

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