1 Repetition

$$t = \frac{T - T_c}{T_c}$$

1.1 critical exponents

exponent	definition magnetic	definition fluid	t	H
α'	$C_H \sim (-t)^{-\alpha'}$	C_v	< 0	0
α	$C_H \sim t^{-\alpha}$	C_V	> 0	0
β	$M \sim (-t)^{\beta}$	$\rho_L - \rho_G$	< 0	0
γ'	$\chi_T \sim (-t)^{-\gamma'}$	κ_T	< 0	0
γ	$\chi_T \sim t^{-\gamma}$	κ_T	> 0	0
δ	$H \sim \mathrm{sgn}(M)\|M\|^{\delta}$	$p - p_c$	0	$\neq 0$
Δ'_l	$\frac{\partial^l G}{\partial H^l} \sim (-t)^{-\Delta'_l)} \frac{\partial^{l-1} G}{\partial H^{(l-1)}}$		< 0	0
Δ_{2l}	$\frac{\partial^{2l} G}{\partial H^{2l}} \sim t^{-2\Delta_{2l})} \frac{\partial^{2l-2} G}{\partial H^{(2l-2)}}$		> 0	0
ν'	$\xi \sim (-t)^{-\nu'}$		< 0	0
ν	$\xi \sim t^{-\nu}$		> 0	0
η	$\Gamma(r) \sim \|r\|^{-(d-2+\eta)}$		0	0

1.2 exponent inequalities

Rushbrooke	$\alpha' + 2\beta + \gamma' \geq 2$
Griffiths	$\alpha' + \beta(1+\delta) \geq 2$
Widom	$\gamma' \geq \beta(\delta - 1)$
	$\ldots$

static scaling hypothesis (Widom 1956) makes statements about the relationship of this exponents. The Kadanoff construction gives explanation why the scaling hypothesis can be applied on Ising model.

1.3 Ising Hamiltonian

$$\begin{aligned} H &= -\sum_{i,j} J'_{ij} s_i s_j - \sum_i h'_i s_i \rightarrow -J' \sum_{(i,j)} s_i s_j - h' \sum_i s_i \\ \beta H &= -J \sum_{(i,j)} s_i s_j - h \sum_{i=1}^{N} s_i \end{aligned}$$

2 Kadanoff construction - Block Spins (1966)

The Idea: At the critical point the correlation length ξ diverges so we can pool a amount of L Ising Spins in every direction to one "Block-Spin", with $L \ll \frac{\xi}{a}$. where a is the lattice constant.

$$
\begin{aligned}
s_\alpha &:= \underbrace{\frac{1}{\left|\sum_{i\in\alpha} s_i\right|}}_{=:c_\alpha^{-1}} \sum_{i\in\alpha} s_i \qquad L \ll \xi/a \\
\beta H &= -\sum_{\alpha=1}^{N/L^d} \sum_{\beta=1}^{N/L^d} \sum_{i\in\alpha, j\in\beta} J_{ij} s_i s_j - h \sum_{\alpha=1}^{N/L^d} \sum_{i\in\alpha}^{N} s_i \\
&= -\sum_{\alpha=1}^{N/L^d} \sum_{\beta=1}^{N/L^d} c_\beta \sum_{i\in\alpha} J_{i\beta} s_i s_\beta - h \sum_{\alpha=1}^{N/L^d} c_\alpha s_\alpha \\
&= -\sum_{(\alpha,\beta)} J_{\alpha\beta} c_\alpha c_\beta s_\alpha s_\beta - h \sum_{\alpha=1}^{N/L^d} c_\alpha s_\alpha
\end{aligned}
$$

$J_{\alpha\beta}$ has again the form of nearest neighbor interaction. Now we assume that for $L \gg 1$ the c_α are normally distributed, so they're all approximately equal. (for $L = 1$ this is exact)

$$
\begin{aligned}
\overset{L\gg 1}{\to} c_\alpha \cong c \Rightarrow \beta H &\cong -\tilde{J} \sum_{(\alpha,\beta)} s_\alpha s_\beta - \tilde{h} \sum_{\alpha=1}^{N/L^d} s_\alpha \\
\tilde{\beta} H &= -\tilde{J} \sum_{(\alpha,\beta)} s_\alpha s_\beta - \tilde{h} \sum_{\alpha=1}^{N/L^d} s_\alpha; \qquad \tilde{\beta} = \frac{1}{k_b \tilde{T}}
\end{aligned}
$$

Now let's have look on the singular part of the Gibbs potential, from which we can get the critical behaviour of the magnetisation. We assume the potential of the Isingspins to have the same form as the potential of the block spins, because the βH have the same form.

$$
\begin{aligned}
\tilde{t} &= \frac{\tilde{T} - T_c}{T_c} \\
\Rightarrow G(t,h) &\sim G(\tilde{t},\tilde{h}) \Rightarrow \bar{G}(t,h) := \frac{G(t,h)}{N} \sim L^{-d} \frac{G(\tilde{t},\tilde{h})}{L^{-d} N} = L^{-d} \bar{G}(\tilde{t},\tilde{h}) \\
\tilde{h} &= f(L)h; \qquad f(L) = L^x \text{ reasonable with two transformations} \\
\tilde{t} &= g(L)t; \qquad g(L) = L^y \\
\bar{G}(L^y t, L^x h) &\sim L^d \bar{G}(t,h); \lambda := L^d \Rightarrow \bar{G}(\lambda^{y/d} t, \lambda^{x/d} h) \sim \lambda \bar{G}(t,h)
\end{aligned}
$$

So G is a generalized homogeneous function, which we can use for

3 static scaling hypothesis (Widom 1956)

If a themodynamical potential is a generalized homogeneous function, we can express all critical exponents with only 2 independent exponents.

$$\begin{aligned}
\bar{G}(\lambda^{y/d}t, \lambda^{x/d}h) &\sim \lambda\bar{G}(t,h) \\
\frac{\partial}{\partial h}\bar{G}(\lambda^{y/d}t, \lambda^{x/d}h) &\sim \frac{\partial}{\partial h}\lambda\bar{G}(t,h) \\
\lambda^{x/d}\frac{\partial}{\partial \lambda^{x/d}h}\bar{G}(\lambda^{y/d}t, \lambda^{x/d}h) &\sim \lambda\frac{\partial}{\partial h}\bar{G}(t,h) \\
\lambda^{x/d}M(\lambda^{y/d}t, \lambda^{x/d}h) &\sim \lambda M(t,h)
\end{aligned}$$

1.$h = 0;\ t \to 0 \Rightarrow \beta$

$$\begin{aligned}
M(t,0) &\sim \lambda^{x/d-1}M(\lambda^{y/d}t, 0) \qquad \text{choose } \lambda = (-t)^{-d/y} \\
&= (-t)^{-x/y+d/y}M(-1,0) \\
&\Rightarrow \beta = \frac{-x+d}{y}
\end{aligned}$$

2.$t = 0;\ h \to 0 \Rightarrow \delta$

$$\begin{aligned}
M(0,h) &\sim \lambda^{x/d-1}M(0, \lambda^{x/d}h) \qquad \text{choose } \lambda = (h)^{-d/x} \\
&= h^{-1+d/x}M(0,1) \Leftrightarrow h \sim M(0,h)^{x/(d-x)} \\
&\Rightarrow \delta = \frac{x}{d-x}
\end{aligned}$$

So we see, how x, y look like

$$\begin{aligned}
x &= \frac{d\delta}{1+\delta} \\
y &= \frac{d-x}{\beta} = \frac{-d\delta + d + d\delta}{(1+\delta)\beta} = \frac{d}{(1+\delta)\beta}
\end{aligned}$$

To get the other exponents relation to x, y and so relations between the critical exponents, we'll need further derivatives of $G(t,h)$

$$\begin{aligned}
\frac{\partial}{\partial h}\lambda^{x/d}M(\lambda^{y/d}t, \lambda^{x/d}h) &\sim \frac{\partial}{\partial h}\lambda M(t,h) \\
\lambda^{2x/d}\frac{\partial}{\partial \lambda^{x/d}h}M(\lambda^{y/d}t, \lambda^{x/d}h) &\sim \lambda\frac{\partial}{\partial h}M(t,h) \\
\lambda^{2x/d-1}\chi(\lambda^{y/d}t, \lambda^{x/d}h) &\sim \chi(t,h) \qquad h = 0; \lambda = (-t)^{-d/y} \\
(-t)^{-2x/y+d/y}\chi(-1,0) &\sim \chi(t,0) \\
&\Rightarrow \gamma' = \frac{2x-d}{y} = \frac{2d\delta - d(1+\delta)}{1+\delta}\frac{(1+\delta)\beta}{d} = (\delta-1)\beta \\
\lambda^{2x/d-1}\chi(\lambda^{y/d}t, \lambda^{x/d}h) &\sim \chi(t,h) \qquad h = 0; \lambda = (t)^{-d/y} \\
t^{-2x/y+d/y}\chi(1,0) &\sim \chi(t,0) \\
&\Rightarrow \gamma = \frac{2x-d}{y} = \gamma'
\end{aligned}$$

$$\begin{aligned}
\lambda\left(\frac{\partial}{\partial h}\right)^l \bar{G}(t,h) &\sim \left(\frac{\partial}{\partial h}\right)^l \bar{G}(\lambda^{y/d}t,\lambda^{x/d}h) \\
\left(\frac{\partial}{\partial h}\right)^l \bar{G}(t,h) &\sim \lambda^{lx/d-1}\left(\frac{\partial}{\partial \lambda^{x/d}h}\right)^l \bar{G}(\lambda^{y/d}t,\lambda^{x/d}h) \\
\frac{\left(\frac{\partial}{\partial h}\right)^l \bar{G}(t,h)}{\left(\frac{\partial}{\partial h}\right)^{l-1} \bar{G}(t,h)} &\sim \lambda^{x/d}\frac{\left(\frac{\partial}{\partial \lambda^{x/d}h}\right)^l \bar{G}(\lambda^{y/d}t,\lambda^{x/d}h)}{\left(\frac{\partial}{\partial \lambda^{x/d}h}\right)^{l-1} \bar{G}(\lambda^{y/d}t,\lambda^{x/d}h)} \to \sim t^{x/y} \\
&\Rightarrow \quad \Delta_l = \frac{x}{y} = \frac{d\delta}{1+\delta}\frac{(1+\delta)\beta}{d} = \beta\delta = \beta+\gamma'
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial}{\partial t}\right)^l \bar{G}(t,h) &\sim \lambda^{ly/d-1}\left(\frac{\partial}{\partial \lambda^{y/d}t}\right)^l \bar{G}(\lambda^{y/d}t,\lambda^{x/d}h) \\
C_h(t,h) &\sim \lambda^{2y/d-1}C_h(\lambda^{y/d}t,\lambda^{x/d}h) \qquad \lambda=(-t)^{-d/y} \\
C_h(t,0) &\sim \lambda^{-2+d/y}C_h(-1,0) \Rightarrow \alpha' = 2-\frac{d}{y} = 2-(1+\delta)\beta \Leftrightarrow \alpha'+2\beta+\gamma' = 2
\end{aligned}$$

4 correlation

To get laws for exponents of correlation depending exponents, we need to have a closer look on the correlation function $\Gamma(r,t)$

$$
\begin{aligned}
s_\alpha &= \frac{1}{c}\sum_{i\in\alpha} = \frac{1}{c'}L^{-d}\sum_{i\in\alpha} s_i \\
\tilde{h} &= c\,h = c'L^d h = L^x h \Rightarrow c' = L^{x-d} \\
\Gamma(r,t) &= \langle (s_i - \langle s_i\rangle)(s_j - \langle s_i\rangle)\rangle\,; \qquad r := \frac{|r_i - r_j|}{a} =: \frac{|r_\alpha + \delta_{\alpha i} - (r_\beta + \delta_{\beta j})|}{a} \\
\Gamma(\tilde{r},\tilde{t}) &= \langle (s_\alpha - \langle s_\alpha\rangle)(s_\beta - \langle s_\beta\rangle)\rangle\,; \qquad \tilde{r} := \frac{|r_\alpha - r_\beta|}{La} \cong \frac{r}{L} \\
&= \left\langle \left(\frac{1}{c'}L^{-d}\sum_{i\in\alpha} s_i - \frac{1}{c'}L^{-d}\sum_{i\in\alpha}\langle s_i\rangle\right)\left(\frac{1}{c'}L^{-d}\sum_{j\in\beta} s_j - \frac{1}{c'}L^{-d}\sum_{j\in\beta}\langle s_j\rangle\right)\right\rangle \\
&= \frac{1}{c'^2}L^{-2d}\sum_{i\in\alpha}^{L^d}\sum_{j\in\beta}^{L^d}\underbrace{\langle (s_i - \langle s_i\rangle)(s_j - \langle s_j\rangle)\rangle}_{\Gamma(r,t)} \\
&\cong \frac{1}{c'^2}\Gamma(r,t) = L^{-2(x-d)}\Gamma(r,t) \\
\Gamma(r,t) &\sim r^{-d+2-\eta}e^{-r/\xi} \text{is Ornstein-Zernicke behaviour of the correlation function} \\
\Gamma(r,t) &= L^{2(x-d)}\Gamma(\tilde{r},\tilde{t}) = L^{2(x-d)}\Gamma(L^{-1}r, L^y t) \qquad L = r, t = 0 \\
\Gamma(r,0) &= r^{2(x-d)}\Gamma(1,0) \Rightarrow -(d-2+\eta) = 2(x-d) \\
&\Leftrightarrow \eta = -2x + d + 2 = \frac{-2d\delta + d(1+\delta) + 2 + 2\delta}{1+\delta} = \frac{d(1-\delta) + 2(1+\delta)}{1+\delta} \\
\Gamma(r,t) &= = L^{2(x-d)}\Gamma(L^{-1}r, L^y t) \qquad L = t^{-1/y} \\
\Gamma(r,t) &= = t^{-2(x-d)/y}\Gamma(t^{1/y}r, 1) \\
\Rightarrow r^{-d+2-\eta}e^{-r/\xi(t)} &\sim t^{-2(x-d)/y}(t^{1/y}r)^{-d+2-\eta}e^{-t^{1/y}r/\xi(1)} \\
e^{-r/\xi(t)} &\sim t^{(-\eta+2-2x+d)/y}e^{-t^{1/y}r/\xi(1)} = e^{-t^{1/y}r/\xi(1)} \\
\xi(t) &\sim t^{-1/y} \Rightarrow \nu = \frac{1}{y} = \frac{(1+\delta)\beta}{d} \\
& \nu' \text{ analogous}
\end{aligned}
$$