Mean field approximation of time-delayed feedback control of noise-induced oscillations in the Van der Pol system

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Abstract. – We study time-delayed feedback control of noise-induced oscillations analytically and numerically for the paradigmatic model of the Van der Pol oscillator under the influence of white noise. We focus on the regime below the Hopf bifurcation where the deterministic system has a stable fixed point and does not exhibit oscillations. Analytical expressions for the power spectral density and the coherence properties of the stochastic delay differential equation in dependence upon noise intensity, delay time, and feedback strength are derived on the basis of a mean field approximation, and are in good agreement with our numerical simulations of the full nonlinear model. Our analytical results elucidate how the correlation time of the controlled stochastic oscillations can be maximized as a function of delay time and feedback strength.

Introduction. – Time delayed feedback as a control mechanism is often used in systems with deterministic chaos [1], where an unstable periodic orbit embedded in a chaotic attractor can be stabilized [2]. The control scheme uses the difference of a system variable $s(t)$ at time t and the same variable at a delayed time, $s(t-\tau)$, to generate a control force which is coupled back to the system. Variants of this scheme have also been studied, e.g. [3–5]. In contrast to control of deterministic chaos, the control of noise-induced phenomena is still an open problem. Recently, a number of methods were suggested for the control of stochastic resonance [6, 7] and of periodically forced [8], multistable [9, 10], or self-oscillating [11] systems in the presence of noise. In contrast to those investigations, a passive self-adaptive method for the control of oscillations induced merely by noise was proposed only recently [12–14]. It uses time-delayed feedback control to change the coherence of the oscillations, and tune their timescale. In the present work we study a generic model for noise-induced oscillations, the Van der Pol (VdP) oscillator, below the Hopf bifurcation, i.e. in the regime where the deterministic system does not oscillate autonomously. It is applicable to a diversity of nonlinear systems. We have been able to obtain analytical results for our problem that go beyond the usual linearization and take into account the nonlinearity by a self-consistent mean field approach. This allows us to predict analytically the dependence of the coherence and the spectral properties upon the noise intensity, the control feedback strength, and the delay time τ, which complements the numerical studies in [12, 13].

Noisy Van der Pol system with time-delayed feedback. – The following dynamic system describes a VdP oscillator with time-delayed feedback control in the presence of noise [12]:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \left(\varepsilon - x^2\right) y - \omega_0^2 x + K\left(y(t-\tau) - y(t)\right) + D\xi(t) \end{aligned} \tag{1}$$

Here ω_0 denotes the natural oscillation frequency and ε is the bifurcation parameter that governs the dynamics of the deterministic system. The term $D\xi(t)$ represents Gaussian white noise with intensity D, $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$. The parameter K is the strength of the control force and τ is the delay time. In the following we fix $\omega_0 = 1$. For $K = 0$, $D = 0$ and $\varepsilon < 0$ the VdP oscillator has a stable fixed point at the origin $(0, 0)$. At $\varepsilon = 0$ a supercritical Hopf bifurcation occurs and for $\varepsilon > 0$ a stable limit cycle exists. In this case the noise term will mainly affect the phase on the limit cycle, and analytical results for this case have been obtained in [11] on the basis of a phase diffusion model. We consider the system slightly below the Hopf bifurcation and fix $\varepsilon = -0.01$. The deterministic system ($D = 0$) does not exhibit self sustained oscillations. However, the introduction of noise ($D > 0$) into the system evokes noisy oscillations with basic period $T_0 \approx \frac{2\pi}{\omega_0}$. In Fig. 1 phase portraits of noise-induced oscillations in the VdP system are shown for different noise intensities. It is clear that the amplitude of the oscillations and therefore the significance of the nonlinear term x^2y gets larger with increasing noise intensity. Since we are dealing with *noise-induced* oscillations, the noise affects the amplitude *and* phase of the oscillations. Therefore in this case a phase diffusion model [11] is not sufficient.

Next, we use time-delayed feedback to control essential features of the oscillations like

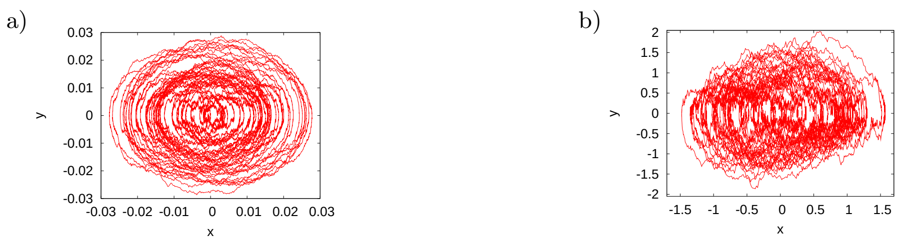


Fig. 1 – Simulated phase portraits of noise-induced oscillations of the Van der Pol system for $\varepsilon = -0.01$, $\omega_0 = 1$, $\tau = 0$, $K = 0$: a) $D = 0.003$; b) $D = 0.5$. Integration time: 300 time units.

timescales and coherence. To quantify the regularity or coherence of the oscillations we introduce the correlation time t_{cor} as [15]

$$t_{cor} = \int_0^\infty |\Psi_{yy}(s)| ds \qquad \text{where} \quad \Psi_{yy}(s) = \frac{1}{\sigma_{yy}^2}\langle(y(t-s) - \langle y\rangle)(y(t) - \langle y\rangle)\rangle \tag{2}$$

is the normalized autocorrelation function of y and $\sigma_{yy}^2 = \langle(y(t) - \langle y\rangle)^2\rangle$, using angular brackets for ensemble average.

The timescales of the oscillations are quantified by the peaks of the power spectral density $S_{yy}(\omega)$, that we will further refer to as spectrum for brevity. The spectrum of the stochastic oscillations is calculated numerically by the Fourier transform of the y variable [16]:

$$S_{yy}(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} \left| \int_0^T y(t) e^{-i\omega t} dt \right|^2 \tag{3}$$

In the following we will study the dependence of t_{cor} and S_{yy} on the control parameters K and τ and on the noise intensity D.

Mean field approximation of the Van der Pol system. – In this section we consider our system without control ($K = 0$). We linearize the system self-consistently with the following mean-field ansatz for the non-linearity:

$$(\varepsilon - x^2) \approx (\varepsilon - \langle x^2 \rangle) \equiv \tilde{\varepsilon} \tag{4}$$

Thus the bifurcation parameter ε is effectively re-scaled. By using this approximation (which becomes exact in the low noise limit $D/(\varepsilon\omega_0) \to 0$),

$$\dot{x} \;=\; y \quad , \qquad \dot{y} = \tilde{\varepsilon} y - \omega_0^2 x + D\xi(t)$$

the VdP system (1) becomes equivalent to a linear stochastic differential equation (SDE), namely a two-dimensional Ornstein-Uhlenbeck process of the form [16]:

$$d\underline{x}_s = -\underline{\underline{A}}\,\underline{x}_s dt + \underline{\underline{B}} d\underline{W}(t) \tag{5}$$

with the constant matrices

$$\underline{\underline{A}} = \begin{pmatrix} 0 & -1 \\ \omega_0^2 & -\tilde{\varepsilon} \end{pmatrix}, \quad \underline{\underline{B}} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \quad \underline{x}_s = (x, y) \tag{6}$$

where the index s stands for "stationary process" (zero mean). From this SDE one can derive an expression for the stationary variance matrix $\underline{\underline{\sigma}} = \langle \underline{x}_s(t), \underline{x}_s^T(t) \rangle$ [16]:

$$\underline{\underline{\sigma}} \;=\; \frac{(\mathrm{Det}\underline{\underline{A}})\underline{\underline{B}}\,\underline{\underline{B}}^T + \left[\underline{\underline{A}} - (\mathrm{Tr}\underline{\underline{A}})\right] \underline{\underline{B}}\,\underline{\underline{B}}^T \left[\underline{\underline{A}} - (\mathrm{Tr}\underline{\underline{A}})\right]^T}{2(\mathrm{Tr}\underline{\underline{A}})(\mathrm{Det}\underline{\underline{A}})} = \frac{D^2}{-2\tilde{\varepsilon}} \begin{pmatrix} \frac{1}{\omega_0^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

where T stands for *transposed*. Now we can calculate $\langle x^2 \rangle$ self-consistently from (7):

$$\sigma_{11} = \langle x^2 \rangle = \frac{D^2}{-2\tilde{\varepsilon}\omega_0^2} = \frac{D^2}{-2(\varepsilon - \langle x^2 \rangle)\omega_0^2} \tag{8}$$

and obtain with (4) and (8) a mean field expression for $\tilde{\varepsilon}$

$$\tilde{\varepsilon} = \frac{\varepsilon}{2}\left(1 + \sqrt{1 + \frac{2D^2}{\varepsilon^2 \omega_0^2}}\right) \tag{9}$$

This is our main result in this section. It shows that the effect of noise in the nonlinear system is to shift the effective bifurcation parameter further away from the Hopf bifurcation. This means the non-linearity enhances the dissipation in our system. Our mean field linearization scheme can also be viewed as a special case of the *statistical linearization* technique [17], however here we use a direct self-consistency condition for ε. Now we are able to find an analytical expression for the correlation time t_{cor}. The time correlation matrix $\underline{\underline{\Psi}}(s)$ of the two-dimensional Ornstein-Uhlenbeck process (5) is given by:

$$\underline{\underline{\Psi}}(s) = \langle \underline{x}_s(t-s), \underline{x}_s^T(t) \rangle = \underline{\underline{\sigma}} \exp\left[-\underline{\underline{A}}^T s\right] = \underline{\underline{\sigma}}\,\underline{\underline{O}} \begin{pmatrix} \exp\lambda_1 s & 0 \\ 0 & \exp\lambda_2 s \end{pmatrix} \underline{\underline{O}}^{-1} \tag{10}$$

where $\underline{\underline{O}}$ is the orthogonal matrix that diagonalizes $\underline{\underline{A}}^T$ and $\lambda_{1,2} = p_{1,2} + iq_{1,2}$ are the eigenvalues of $-\underline{\underline{A}}$. This result is only valid for $p_{1,2} < 0$. In our system (6) we find

$$p_{1,2} \equiv p = \frac{\tilde{\varepsilon}}{2} < 0 \quad , \qquad q_{1,2} \equiv \pm\tilde{\omega} = \pm\sqrt{-\frac{\tilde{\varepsilon}^2}{4} + \omega_0^2} \tag{11}$$

From (10), (11) it follows that the autocorrelation function Ψ_{yy} is of the form $\Psi_{yy}(s) \approx e^{ps}\cos(\tilde{\omega}s)$. With the definition of t_{cor} in (2), and using (11), we find [14]

$$t_{cor} = \int_0^\infty |\Psi_{yy}(t)|dt = \int_0^\infty e^{ps}|\cos(\tilde{\omega}s)|ds \approx \frac{2}{\pi}\int_0^\infty e^{ps}ds = -\frac{2}{\pi p} = -\frac{4}{\pi\tilde{\varepsilon}} \qquad (12)$$

To simplify our calculations, we have used $|\tilde{\varepsilon}| \ll \tilde{\omega}$ and substituted the cosine term by the filling factor $\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(\varphi)d\varphi = \frac{2}{\pi}$. Eq. (12) together with (9) describes the correlation time as a function of D for $K = 0$. In Fig. 2a), middle curve, we see that this result is in excellent agreement with numerical simulations of the nonlinear VdP system over a large range of noise intensities. For $D \to 0$ ($\tilde{\varepsilon} = \varepsilon$) the mean field approximation becomes equal to the

a) b)

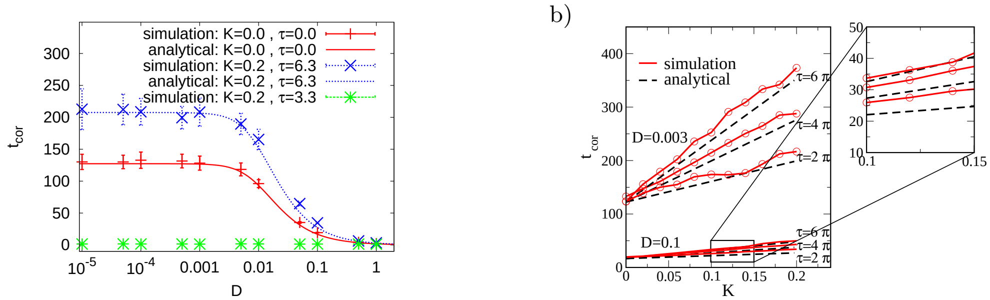


Fig. 2 – Correlation time t_{cor} in the VdP system for $\varepsilon = -0.01$ a) vs. noise intensity D for different values of τ and K (symbols: numerical solution; solid line - analytical mean field estimate from eq. (12) for $K = 0$, and from eq. (18) for $K = 0.2, \tau = 6.3$; b) vs. feedback strength K for three different values of τ; analytical - from eq. (18)

usual linearization [14] which neglects the dependence upon the noise intensity D. Hence the mean field model is essential in explaining the drop in the correlation time t_{cor} for large noise intensities ($D > 0.001$), see Fig. 2a),b), 3a).

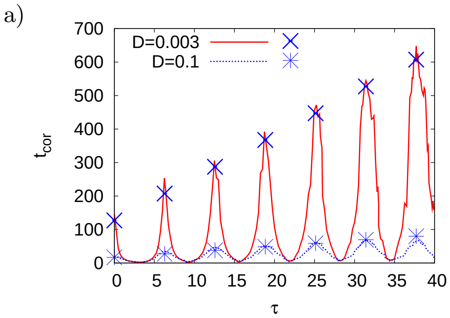
Correlation time of the controlled Van der Pol system. – We shall now discuss the system with time-delayed feedback control. Below the Hopf bifurcation, the noise-induced oscillations of the VdP system occur in the vicinity of the fixed point $(0, 0)$, see Fig. 1a). Hence the oscillation properties should be governed by the local stability of the fixed point. We use the mean-field approximation (4) for $\varepsilon - x^2$, which goes beyond the conventional linearization around $(0, 0)$ [12–14], but immediately allows for a linear stability analysis. Including delayed-feedback control but neglecting the noise term, we can rewrite the system as a single second order delay differential equation:

$$\ddot{x} - \tilde{\varepsilon}\dot{x} + \omega_0^2 x - K(\dot{x}(t-\tau) - \dot{x}(t)) = 0 \qquad (13)$$

Using the exponential ansatz $x \propto e^{\lambda t}$ we get the characteristic equation for the complex eigenvalues $\lambda = p + iq$:

$$\lambda^2 - \tilde{\varepsilon}\lambda + \omega_0^2 - K\lambda\left(e^{-\lambda\tau} - 1\right) = 0 \qquad (14)$$

This is a transcendental equation for the eigenvalues λ. The system becomes infinite dimensional due to the delay term and we obtain a countable set $\lambda_j^e = p_j^e + iq_j^e$, $j = 1, 2, \ldots,$ of eigenvalues. The characteristic equation can be solved numerically, see [12–14]. For a better comparison with the delay time τ, it is also convenient to consider the eigenperiods $T_i^e = \frac{2\pi}{q_i^e}$ instead of the imaginary parts q_i^e of λ_i^e. For the delay time $\tau \approx n\frac{2\pi}{\omega_0}$ close to integer mul-



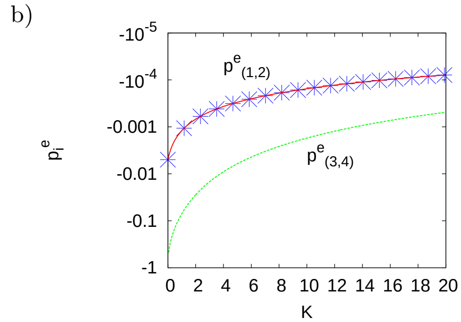


Fig. 3 – a) Correlation time t_{cor} in the VdP system vs. τ for $\varepsilon = -0.01$, $K = 0.2$. Solid line - numerical simulations, symbols - analytical estimate by (18); b) Largest real parts of characteristic equation in dependence on K for $\tilde{\varepsilon} = -0.01$, $\tau = 2\pi$. Solid lines - from numerical solution of characteristic equation (14), symbols - real part δ_p from linearized characteristic equation (17)

tiples of the basic period $T_0 \approx 2\pi/\omega_0$ two complex conjugate eigenvalues are of the form $\lambda^e_{1,2} = \delta_p \pm i(1+\delta_q)\omega_0$ with $|\delta_p|,|\delta_q| \ll 1$ since $T^e_{1,2} \approx \pm\frac{2\pi}{\omega_0}$. These eigenvalues have a real part close to zero, and the other eigenvalues λ^e_i $(i > 2)$ are far away from the real axis [12–14]. In fig. 3b) we see that this is valid over a large range of the feedback strength K, for $\tau = 2\pi$ the largest real parts $p^e_{(1,2)}$ are separated by approximately one order of magnitude from the second largest real parts $p^e_{(3,4)}$. This situation is comparable to the limit $\tau \to 0$ which is the Ornstein-Uhlenbeck process. There we have also two complex conjugate eigenvalues with real part $p^e_{1,2} = p$ close to zero, see eq. (11), and the others have real parts with $p^e_i \to -\infty$ $(i > 2)$, i.e. they are far away from the real axis. Because of that we will approximate the correlation time t_{cor} for the case with control and $\tau \approx n\frac{2\pi}{\omega_0}$ by

$$t_{cor} = -\frac{2}{\pi\delta_p} \tag{15}$$

in analogy with eq. (12) for the Ornstein-Uhlenbeck process. This means we restrict our analysis to the two least stable eigenvalues $\lambda^e_{1,2}$. Below we will see that the regularity of the noisy oscillations has maxima for τ close to multiples of the basic period and thereby derive analytical expressions for δ_p and δ_q. We will thus find an analytical expression for t_{cor} using (15). We linearize the characteristic equation (14) in δ_p and δ_q neglecting terms of the order $O(\delta_p^2)$, $O(\delta_q^2)$, $O(\delta_p\delta_q)$ and $O(\delta_p\tilde{\varepsilon})$, $O(\delta_q\tilde{\varepsilon})$ since $\tilde{\varepsilon} \ll 1$. We arrive at the linearized characteristic equation:

$$(2i\omega_0\Lambda - i\tilde{\varepsilon}\omega_0) + Ki\omega_0\left(1 - e^{-i\omega_0\tau} + e^{-i\omega_0\tau}\tau\Lambda\right) + K\Lambda\left(1 - e^{-i\omega_0\tau}\right) = 0 \tag{16}$$

with $\Lambda \equiv \delta_p + i\delta_q\omega_0$. For $\tau = n\frac{2\pi}{\omega_0}$ we find from (16)

$$\delta_q = 0 \quad , \qquad \delta_p = \frac{\tilde{\varepsilon}}{2}\frac{1}{1+\frac{K}{2}\tau} \tag{17}$$

Note that in the derivation of (17) we have used that δ_p and δ_q are small. This would be violated, for example, for $K \approx -2/\tau$. Here we restrict ourselves to the case of $K > 0$, where (17) holds. In Fig. 3b) this result for the real part δ_p from the linearized characteristic equation (16) is shown in comparison to the largest real part of the nonlinear characteristic equation (14). They are in excellent agreement. Substituting (17) into (15) yields the correlation time

$$t_{cor} = -\frac{4}{\pi\tilde{\varepsilon}}\left(1 + \frac{K}{2}\tau\right) \tag{18}$$

This result is valid for $\tau = n\frac{2\pi}{\omega_0}$ and, in the low-noise limit $D \to 0$, $\tilde{\varepsilon} \to \varepsilon$, reduces to the expression for t_{cor} in the case of optimal control as derived in [14] via the spectrum. In Figs. 2a),b) and 3a) we see that our analytical result is in very good agreement with numerical simulations of the nonlinear system. Eq. (18) gives a quantitative prediction how the coherence depends upon D, K, and τ. For optimum $\tau = n\frac{2\pi}{\omega_0}$ the coherence is enhanced by the time-delayed feedback by a factor of $(1 + \frac{K}{2}\tau)$. The coherence is modulated as a function of τ, and attains minima close to zero for $\tau = (2n-1)\frac{\pi}{\omega_0}$.

Analytical approximation of the spectrum. – Next we derive an analytical expression for the spectrum of the VdP system. Using the linear mean field approximation (4) we can Fourier transform eq. (1): $-i\omega\hat{x}(\omega) = \hat{y}(\omega)$, $-i\omega\hat{y}(\omega) = \tilde{\varepsilon}\hat{y}(\omega) - \omega_0^2\hat{x}(\omega) + D\hat{\xi}(\omega) + K\hat{y}(e^{i\omega\tau} - 1)$ Taking into account that $\langle\hat{y}(\omega)^*\hat{y}(\omega')\rangle = \delta(\omega - \omega')S_{yy}(\omega)$ and $\langle\hat{\xi}(\omega)\hat{\xi}^*(\omega')\rangle = \frac{1}{2\pi}\delta(\omega - \omega')$ we find for the spectrum $S_{yy}(\omega)$:

$$S_{yy}(\omega) = \frac{D^2}{2\pi}\frac{\omega^2}{(\omega^2 - \omega_0^2 + \omega K\sin(\omega\tau))^2 + \omega^2(\tilde{\varepsilon} - K(1 - \cos(\omega\tau))^2} \tag{19}$$

Unlike the conventional linear approximation used in [14], this mean field approximation is in excellent agreement with numerical simulations of the full nonlinear system even for large noise intensities D, see Fig. 4. For $K = 0$, the spectrum is a Lorentzian with peak frequency $\approx \omega_0$ and approximate half-width $-\tilde{\varepsilon}/2 = \pi/(2\,t_{cor})$; for $K \neq 0$ an increasing number of additional peaks appears with increasing τ. For $D = 0.003$ (low noise intensity, Fig. 4a) the mean field approximation is almost equal to the usual linearization ($\tilde{\varepsilon} \approx 1.04\varepsilon$, Eq. (9)), cf. [14]. For $D = 0.5$ (high noise intensity), however, we have $\tilde{\varepsilon} \approx 36\varepsilon$, and the spectrum differs substantially. In order to visualize the effect of the mean-field approximation we keep all parameters constant except for the noise intensity (Fig. 4b). In Fig. 4c) the analytical result with and without ($\tilde{\varepsilon} = \varepsilon$) the mean field approximation are compared to a numerical simulation. The mean field approximation reproduces all details of the changes in the spectrum very well.

Control of noise induced oscillations. – Fig. 3a) shows that time-delayed feedback can be used to control the coherence of oscillations very effectively. For optimal values of $\tau \approx n\frac{2\pi}{\omega_0} = nT_0$ the regularity of the oscillations has maxima. These maxima become larger with increasing delay time τ (i.e., n) and feedback strength K, see fig. 2b). The result is in good agreement with our analytical approximation (18). Fig. 2a) shows that time-delayed feedback can be used over a large range of noise intensities. For τ between optimal values time-delayed feedback can be used to destroy the regularity of oscillations. Furthermore the timescales (i.e., frequency) of the oscillations can be controlled by time-delayed feedback. The period T_s (i.e., inverse of frequency) of the largest peak in the spectrum can be modified by varying τ around its optimum values $\tau = nT_0$ leading to a piecewise linear dependence upon τ [12]. The frequency of the largest peak corresponds to the eigenperiod T^e_{max} with the largest corresponding real part $p_{max} < 0$, i.e. the one with the smallest modulus. This means that the least stable eigenmodes introduced by the time-delay can be excited most easily by noise. Maxima of the real parts p^e_i correspond to maxima of the correlation time t_{cor} (Fig. 3a). The regularity of the oscillations becomes larger, the less stable an eigenmode is. For optimal values of τ the correlation time t_{cor} depends linearly on the feedback strength K (Fig. 2b).

Conclusion. – Using the VdP system as a paradigm of nonlinear oscillators, we have shown that the control of coherence and spectral properties of noise-induced oscillations by a time-delayed feedback scheme can be well understood analytically. To this purpose we have

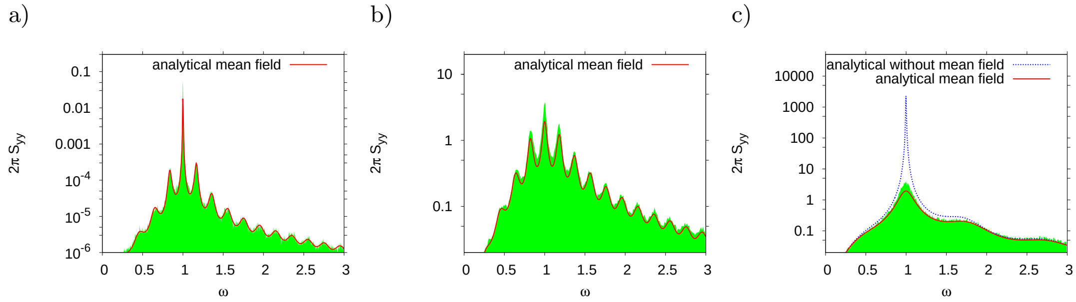


Fig. 4 – Spectrum $S_{yy}(\omega)$ for the VdP system in the presence of delayed feedback for $\varepsilon = -0.01$, $K = 0.2$: a) $D = 0.003$, $\tau = 31.4$; b) $D = 0.5$, $\tau = 31.4$; c) $D = 0.5$, $\tau = 6.3$; Shaded - numerically simulated spectra, solid line - spectrum estimated analytically by (19), dashed line in c) - spectrum estimated analytically without mean field ($\tilde{\varepsilon} = \varepsilon$) [14].

developed a mean-field approximation of the nonlinear system, which takes into account the nonlinearity self-consistently and goes beyond the usual linearization. The mean field model describes the functional dependence of the correlation time and the power spectral density upon the noise intensity D, the control feedback strength K, and the delay time τ in a wide range of parameters in excellent agreement with numerical simulations. It provides a good approximation even in the case of quite large noise and applied control. By adjusting the control parameters τ and K we are able to modify essential oscillation features like coherence and timescales.

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