

TIME-DELAY FEEDBACK CONTROL OF NONLINEAR STOCHASTIC OSCILLATIONS

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Abstract

Effects of time-delay feedback on noisy dynamics of nonlinear oscillators are studied both analytically and numerically. It is shown that such a feedback in the form proposed earlier by Pyragas for the control of chaotic oscillations can be used for effective manipulation of the statistics of noisy oscillations either above or below the Hopf bifurcation of the deterministic dynamic system; however, the action of the feedback in those two cases is different. As a particular, paradigmatic model we analyse a Van der Pol oscillator under the influence of white noise in the regime below the Hopf bifurcation where the deterministic system has a stable fixed point. We show that both the coherence and the frequency of the noise-induced oscillations can be controlled by varying the delay time and the strength of the control force. Approximate analytical expressions for the power spectral density and the coherence properties of the stochastic delay differential equation are developed, and are in good agreement with our numerical simulations. Our analytical results elucidate how the correlation time of the controlled stochastic oscillations can be maximized as a function of delay and feedback strength.

Key words

stochastic delay-differential equations, noise-induced oscillations, control

1 Introduction

Time delayed feedback as a control mechanism is often used in systems with deterministic chaos [Pyragas, 1992], where an unstable periodic orbit embedded in a chaotic attractor can be stabilized [Schuster, 1999; Un-

kelbach, 2003]. The delayed feedback strength K is multiplied with the difference of a signal s from the system at times t and $t - \tau$ where t is the current time and τ is a time delay. In the present work we study time-delayed feedback control of a stochastic system. We investigate a generic model of a nonlinear oscillator, the Van der Pol oscillator, below and above the Hopf bifurcation under the influence of Gaussian white noise. Although the noisy oscillations in the two different regimes have a different origin we try to control them with the same method, time-delayed feedback. Our model is described by a nonlinear stochastic delay differential equation. The theory of these equations is still under development, although some recent studies have focussed upon stochastic systems under control [Landa, 1997; Gammaitoni, 1999; Lindner, 2001; Tsimring, 2001; Masoller 2002; Goldobin, 2003]. We have been able to obtain analytical results for our problem that go beyond the usual linearization and take into account the nonlinearity by a self-consistent mean field approach.

2 Noisy Van der Pol system with time-delayed feedback

The following dynamic system describes a Van der Pol oscillator with time-delayed feedback in the presence of noise:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= (\epsilon - x^2)y - \omega_0^2 x + K(y(t - \tau) - y(t)) + D\xi(t)\end{aligned}\tag{1}$$

Here ω_0 denotes the natural oscillation frequency and ϵ is the bifurcation parameter that governs the dynam-

ics of the system. The term $D\xi(t)$ represents Gaussian white noise with zero mean and intensity defined by the parameter D .

$$\begin{aligned}\langle \xi(t) \rangle &= 0 \\ \langle \xi(t)\xi(t') \rangle &= \delta(t-t')\end{aligned}\quad (2)$$

The parameter K is called delayed feedback strength and τ is the delay time. In the following we fix $\omega_0 = 1$. For $K = 0$, $D = 0$ and $\epsilon < 0$ the Van der Pol oscillator has a stable fixed point at the origin $(0, 0)$. At $\epsilon = 0$ a Hopf bifurcation occurs and for $\epsilon > 0$ a stable limit cycle in the phase plane exists.

Introduction of noise into the system has different effects in the two different regimes ($\epsilon \leq 0$). Below the Hopf bifurcation ($\epsilon < 0$) the system does not exhibit self-sustained oscillations. However, the introduction of noise ($D > 0$) into the system evokes noisy oscillations with basic period $T_0 \approx \frac{2\pi}{\omega_0}$, see Fig. 1(a). Above the Hopf bifurcation the system oscillates even without noise ($D = 0$) with approximately the same basic period T_0 . Here the introduction of noise smears out the stable limit cycle in the phase plane, see Fig. 1(b). In spite of the different regimes the phase por-

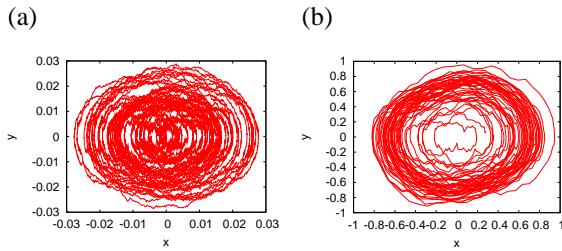


Figure 1. Numerically simulated phase portraits of noisy oscillations of the Van der Pol system for $\omega_0 = 1$, $\tau = 0$, $K = 0$:

(a) below Hopf bifurcation $\epsilon = -0.01$, $D = 0.003$;

(b) above Hopf bifurcation $\epsilon = 0.1$, $D = 0.1$

In both cases the system was integrated during 300 time units.

traits look very similar. Therefore we use time-delayed feedback in both cases to control essential oscillation features like timescales and coherence. To quantify the regularity or coherence of the oscillations we introduce the correlation time t_{cor} as [Stratonovich, 1963]

$$t_{cor} = \int_0^\infty |\Psi_{yy}(s)| ds \quad (3)$$

where

$$\Psi_{yy}(s) = \frac{1}{\sigma_{yy}^2} \langle (y(t-s) - \langle y \rangle)(y(t) - \langle y \rangle) \rangle \quad (4)$$

is the normalized autocorrelation function of y and $\sigma_{yy}^2 = \langle (y(t) - \langle y \rangle)^2 \rangle$.

The timescales of the oscillations are quantified by the power spectral density $S_{yy}(\omega)$, that we will further refer to as spectrum for brevity. The spectrum of the stochastic oscillations is calculated numerically by the Fourier transform of the y variable of the stochastic process (1). The exact definition is [Gardiner, 2002]:

$$S_{yy}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \left| \int_0^T y(t) e^{-i\omega t} dt \right|^2 \quad (5)$$

In the following we will study the dependence of t_{cor} and S_{yy} on the control parameters K and τ and on the noise intensity D .

3 Control of noise-induced oscillations below the Hopf bifurcation

In this section we will look at the Van der Pol system slightly below the Hopf bifurcation (we fix $\epsilon = -0.01$) [Janson, 2004; Balanov, 2004; Schöll, 2005]. We will derive analytical expressions for the correlation time and the spectrum by linearization and a mean field approximation of (1) and compare them to numerical simulations.

3.1 Mean field approximation of the Van der Pol system

In this section we consider our system without control ($K = 0$). We linearize our system self-consistently with the following ansatz for the non-linearity:

$$(\epsilon - x^2) \approx (\epsilon - \langle x^2 \rangle) =: \tilde{\epsilon} \quad (6)$$

By using this approximation the Van der Pol system (1) becomes equivalent to a linear stochastic differential equation (SDE) namely a two-dimensional Ornstein-Uhlenbeck process of the form [Gardiner, 2002]:

$$d\underline{x}_s = -\underline{A}\underline{x}_s dt + \underline{B}dW(t) \quad (7)$$

in our case we have ($K = 0$)

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \tilde{\epsilon}y - \omega_0^2 x + D\xi\end{aligned}\quad (8)$$

and therefore get the constant matrices

$$\underline{A} = \begin{pmatrix} 0 & -1 \\ \omega_0^2 & -\tilde{\epsilon} \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \quad \underline{x}_s = (x, y) \quad (9)$$

where index s stands for "stationary". From this SDE one can derive an expressions for the stationary vari-

ance matrix [Gardiner, 2002]:

$$\begin{aligned}\underline{\underline{\sigma}} &= \langle \underline{x}_s(t), \underline{x}_s^T(t) \rangle \\ &= \frac{(\text{Det}A)BB^T + [A - (\text{Tr}A)]BB^T[A - (\text{Tr}A)]^T}{2(\text{Tr}A)(\text{Det}A)} \\ &= \frac{D^2}{-2\tilde{\epsilon}} \begin{pmatrix} \frac{1}{\omega_0^2} & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}\quad (10)$$

where T stands for *transposed*. Now we can calculate $\langle x^2 \rangle$ self-consistently from (10):

$$\sigma_{11} = \langle x^2 \rangle = \frac{D^2}{-2\tilde{\epsilon}\omega_0^2} = \frac{D^2}{-2(\epsilon - \langle x^2 \rangle)\omega_0^2}\quad (11)$$

and get with (6) and (11) a mean field expression for $\tilde{\epsilon}$

$$\tilde{\epsilon} = \frac{\epsilon}{2} \left(1 + \sqrt{1 + \frac{2D^2}{\epsilon^2\omega_0^2}} \right)\quad (12)$$

Now we want to find an analytical expression for the correlation time t_{cor} . The time correlation matrix $\underline{\underline{\Psi}}(s)$ of the two-dimensional Ornstein-Uhlenbeck process (7) is given by:

$$\begin{aligned}\underline{\underline{\Psi}}(s) &= \langle \underline{x}_s(t-s), \underline{x}_s^T(t) \rangle \\ &= \underline{\underline{\sigma}} \exp[-\underline{\underline{A}}^T s] \\ &= \underline{\underline{\sigma}} \underline{\underline{Q}} \begin{pmatrix} \exp \lambda_1 s & 0 \\ 0 & \exp \lambda_2 s \end{pmatrix} \underline{\underline{Q}}^{-1}\end{aligned}\quad (13)$$

where $\underline{\underline{Q}}$ is the matrix that diagonalizes $\underline{\underline{A}}^T$ and $\lambda_{1,2} = p_{1,2} + iq_{1,2}$ are the eigenvalues of $-\underline{\underline{A}}$. This result is only valid for $p_{1,2} < 0$. In our system (9) we find

$$p_{1,2} \equiv p = \frac{\tilde{\epsilon}}{2} < 0\quad (14)$$

$$q_{1,2} \equiv \pm \tilde{\omega} = \pm \sqrt{-\frac{\tilde{\epsilon}^2}{4} + \omega_0^2}\quad (15)$$

We then see that with (13), (14) and (15) that the correlations Ψ_{yy} are of the form:

$$\Psi_{yy}(s) \approx e^{ps} \cos(\tilde{\omega}s)\quad (16)$$

With the definition of t_{cor} (3) we find [Schöll, 2005]

$$\begin{aligned}t_{cor} &= \int_0^\infty |\Psi_{yy}(t)| dt \\ &= \int_0^\infty e^{ps} |\cos(\tilde{\omega}s)| ds \\ &\approx \frac{2}{\pi} \int_0^\infty e^{ps} ds \\ &= -\frac{2}{\pi p}\end{aligned}\quad (17)$$

$$\stackrel{(14)}{=} -\frac{4}{\pi \tilde{\epsilon}}\quad (18)$$

To simplify our calculations we have used that $|\tilde{\epsilon}| \ll \tilde{\omega}$ and substituted the cos term by the filling factor $\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\varphi) d\varphi = \frac{2}{\pi}$. In Fig. 2 we see that this result is in excellent agreement with numerical simulations over a large range of noise intensities.

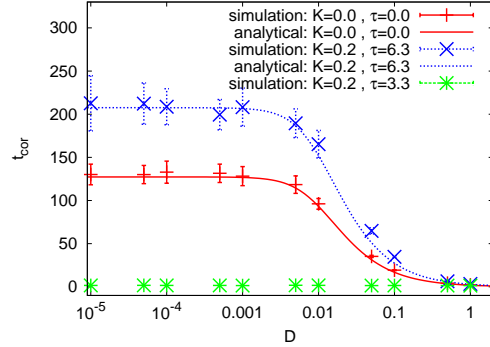


Figure 2. Correlation time t_{cor} of noise-induced oscillations in the Van der Pol system in dependence on noise intensity D for $\omega_0 = 1$, $\epsilon = -0.01$; solid line - analytical mean field estimated by (18) for $K = 0.0$, $\tau = 0.0$ and (26) for $K = 0.2$, $\tau = 6.3$

3.2 Linear stability analysis of the Van der Pol system

Below the Hopf bifurcation the noise-induced oscillations of the Van der Pol system occur in the vicinity of the fixed point, see Fig. 1(a). Hence the oscillation properties should be governed by the stability of the fixed point [Janson, 2004; Balanov, 2004]. Therefore we consider the linearized Van der Pol system including delayed-feedback control but without noise ($D = 0$). We use our mean field approximation for $\epsilon - x^2$ (6). We rewrite the system as a single second order delay differential equation:

$$\ddot{x} - \tilde{\epsilon}\dot{x} + \omega_0^2 x - K(\dot{x}(t-\tau) - \dot{x}(t)) = 0\quad (19)$$

Using the exponential ansatz $x \propto e^{\lambda t}$ we get the characteristic equation for the complex eigenvalues $\lambda = p + iq$:

$$\lambda^2 - \tilde{\epsilon}\lambda + \omega_0^2 - K\lambda(e^{-\lambda\tau} - 1) = 0\quad (20)$$

This is a transcendental equation for the eigenvalues λ . The system becomes infinite dimensional due to the delay term and we get a countable set $\lambda_j^e = p_j^e + iq_j^e$ of eigenvalues. The characteristic equation can be solved numerically, see Fig. 3 (a),(b). We show the eigenperiod $T_i^e = \frac{2\pi}{q_i^e}$ instead of the imaginary part q_i^e of λ_i^e . That way we can compare it better to the delay time τ .

We see that for the delay time $\tau \approx n \frac{2\pi}{\omega_0}$ close to integer multiples of the basic period one eigenvalue is of the form $\lambda = \delta_p \pm i(1 + \delta_q)\omega_0$ with $|\delta_p|, |\delta_q| \ll 1$ since $T_i^e \approx \frac{2\pi}{\omega_0}$, see Fig. 3(a). This means one eigenvalue has

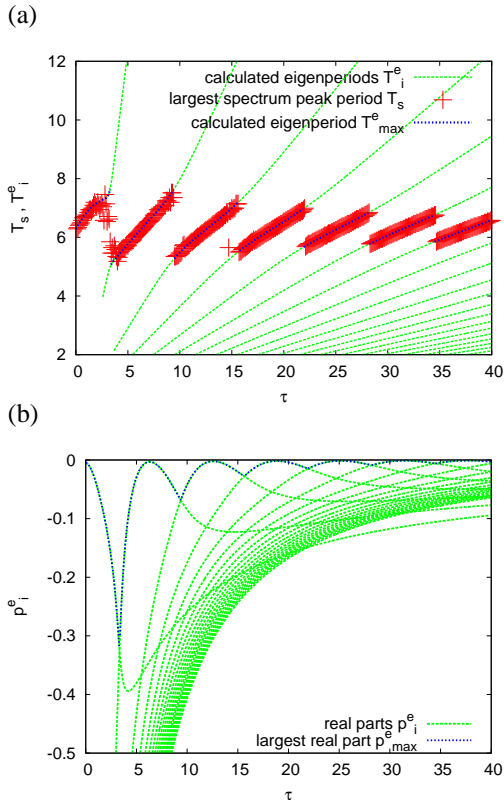


Figure 3. (a) Largest spectrum peak periods T_s , eigenperiods $T_i^e = \frac{2\pi}{q_i}$ of characteristic equation (20), eigenperiods T_{max}^e corresponding to largest real part p_{max} (b) real parts p_i^e and largest real parts p_{max}^e of characteristic equation (20)

Parameters: $\omega_0 = 1$, $\epsilon = -0.01$, $K = 0.2$

a real part close to 0 and the other eigenvalues are far away from the real axis. This situation is comparable to the case $\tau \rightarrow 0$ which is the Ornstein-Uhlenbeck process. There we have one eigenvalue with real part close to 0 and the others have real parts with $p_i^e \rightarrow -\infty$. Because of that we will approximate the correlation time t_{cor} by

$$t_{cor} = -\frac{2}{\pi\delta_p} \quad (21)$$

like in the last section (17) for the Ornstein-Uhlenbeck. Later we will see that the regularity of the noisy oscillations has maxima for τ close to multiples of the basic period and therefore derive an analytical expressions for δ_p and δ_q . Doing that we find an analytical expression for t_{cor} using (21). We linearize the characteristic equation (20) in δ_p and δ_q neglecting terms of the order $O(\delta_p^2)$, $O(\delta_q^2)$, $O(\delta_p\delta_q)$ and $O(\delta_p\tilde{\epsilon})$, $O(\delta_q\tilde{\epsilon})$ since $\tilde{\epsilon} \ll 1$. We arrive at the linearized characteristic equation:

$$(2i\omega_0\Lambda - i\tilde{\epsilon}\omega_0) + Ki\omega_0(1 - e^{-i\omega_0\tau} + e^{-i\omega_0\tau}\tau\Lambda) + K\Lambda(1 - e^{-i\omega_0\tau}) = 0 \quad (22)$$

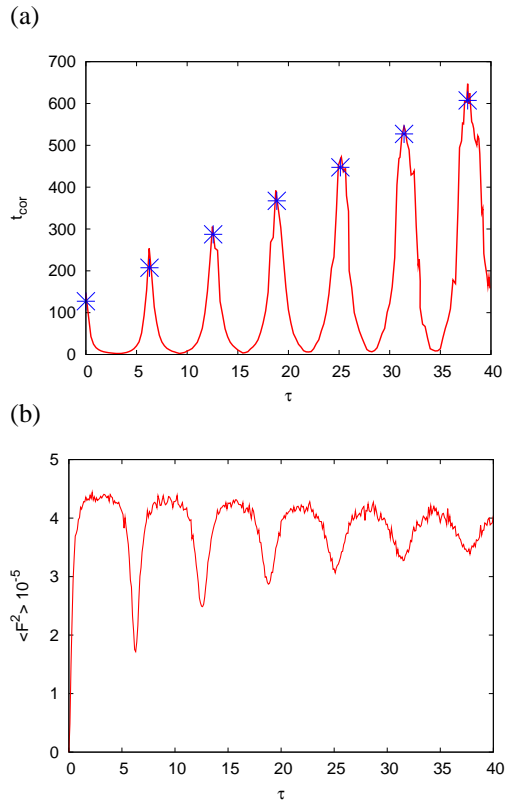


Figure 4. (a) Correlation time t_{cor} of noise-induced oscillations in the Van der Pol system in the presence of delayed feedback in dependence on τ for $D = 0.003$

solid line - t_{cor} from numerical simulations, points - t_{cor} estimated analytically by (26)

(b) Second moment $\langle F^2 \rangle$ of the control force F in dependence on τ for $D = 0.003$ Parameters: $\omega_0 = 1$, $\epsilon = -0.01$, $K = 0.2$

with

$$\Lambda = \delta_p + i\delta_q\omega_0 \quad (23)$$

For $\tau = n\frac{2\pi}{\omega_0}$ we find from (22)

$$\delta_q = 0 \quad (24)$$

$$\delta_p = \frac{\tilde{\epsilon}}{2} \frac{1}{1 + \frac{K}{2}\tau} \quad (25)$$

Now we relate this result to the correlation time using (21) and (25)

$$t_{cor} = -\frac{4}{\pi\tilde{\epsilon}} \left(1 + \frac{K}{2}\tau\right) \quad (26)$$

This result is valid for $\tau = n\frac{2\pi}{\omega_0}$ and matches the expression for t_{cor} in the case of optimal control derived in [Schöll, 2005] via the spectrum. In Fig. 2, 4(a) and 5 we see that our result is in very good agreement with numerical simulations.

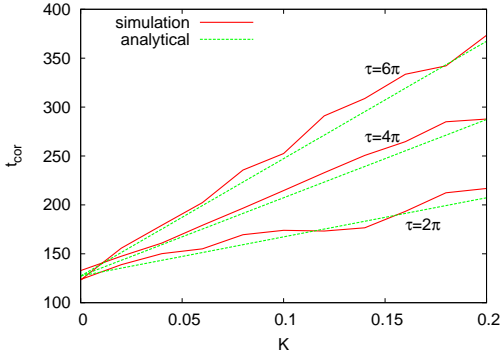


Figure 5. Correlation time t_{cor} of noise-induced oscillations in the Van der Pol system in dependence on feedback strength K for three different values of τ , $\omega_0 = 1$, $\epsilon = -0.01$, $D = 0.003$; analytical curve estimated by (26)

3.3 Analytical approximation of the spectrum

Next we derive an analytical expression for the spectrum of the Van der Pol system (1) [Schöll, 2005]. We linearize our system by using the mean field approximation (6) and Fourier transform it:

$$\begin{aligned} -i\omega\hat{x}(\omega) &= \hat{y}(\omega) \\ -i\omega\hat{y}(\omega) &= \tilde{\epsilon}\hat{y}(\omega) - \omega_0^2\hat{x}(\omega) + D\hat{\xi}(\omega) + K\hat{y}(e^{i\omega\tau} - 1) \end{aligned} \quad (27)$$

Taking into account that

$$\langle \hat{y}(\omega)^* \hat{y}(\omega') \rangle = \delta(\omega - \omega') S_{yy}(\omega) \quad (28)$$

and using $\hat{y}(\omega) = -i\omega\hat{x}(\omega)$ and $\langle \hat{\xi}(\omega)\hat{\xi}^*(\omega') \rangle = \frac{1}{2\pi}\delta(\omega - \omega')$ we find for the spectrum $S_{yy}(\omega)$:

$$S_{yy}(\omega) = \frac{D^2}{2\pi} \frac{\omega^2}{(\omega^2 - \omega_0^2 + K\omega \sin(\omega\tau))^2 + \omega^2(\tilde{\epsilon} - K(1 - \cos(\omega\tau)))^2} \quad (29)$$

Since we used the mean field approximation this analytical expression can also be used at large noise intensities. We see that our analytical approximation is in excellent agreement with numerical simulations even for large noise intensities D , Fig. 6(a),(b).

3.4 Control of noise induced oscillations

Fig. 4(a) shows that time-delayed feedback can be used to control the coherence of oscillations very effectively. For optimal values of $\tau \approx n \frac{2\pi}{\omega_0} = nT_0$ the regularity of the oscillations has maxima. These maxima become larger with increasing delay time τ (respectively n) and feedback strength K . The result is in good agreement with our analytical approximation (26). Fig. 2 shows that time-delayed feedback can be used over a large range of noise intensities. For τ between optimal values time-delayed feedback can be used to destroy the regularity of oscillations. Furthermore the timescales (i.e. the frequency) of the oscillations can be controlled by time-delayed feedback. In Fig. 3(a)

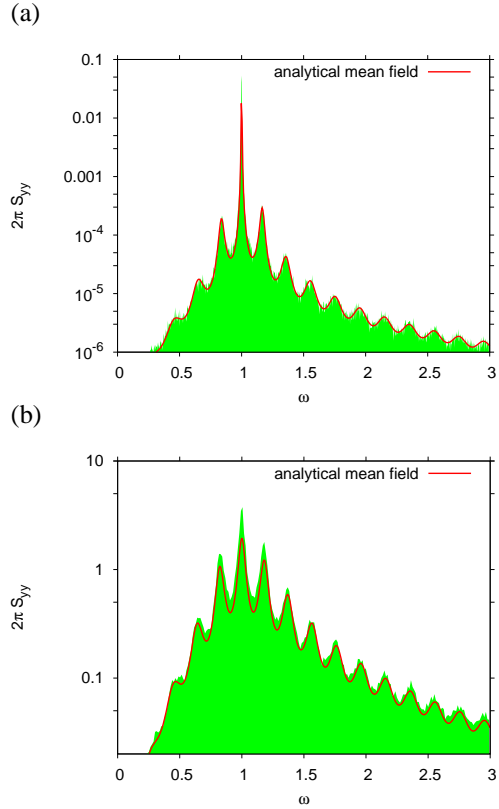


Figure 6. Spectrum $S_{yy}(\omega)$ of noise-induced oscillations in the Van der Pol system in the presence of delayed feedback for $\omega_0 = 1$, $\epsilon = -0.01$, $\tau = 31.4$, $K = 0.2$:

(a) low noise intensity $D = 0.003$;

(b) large noise intensity $D = 0.5$

Shaded - numerically simulated spectra, solid line - spectra estimated analytically by (29).

we see that the period T_s (inverse of frequency) of the largest peak in the spectrum can be modified by varying τ . The frequency of the largest peak corresponds to the eigenperiod T_{max}^e with the largest corresponding real part p_{max} . This means that the least stable eigenmodes introduced by the time-delay can be excited most easily by noise. Maxima of the real parts p_i^e correspond to maxima of the correlation time t_{cor} , Fig. 3(b), 4(a). The regularity of the oscillations becomes larger the less stable an eigenmode is. For optimal values of τ the correlation time t_{cor} depends linearly on the feedback strength K , Fig. 5. Next we look at the force that is needed to control the Van der Pol system. We define the control force F

$$F = y(t - \tau) - y(t) \quad (30)$$

and look at its second moment $\langle F^2 \rangle$. In Fig. 4(b) we see that the control force has minima for optimal values of τ . This means less control force is needed to control more regular behaviour.

4 Control of noisy oscillations above the Hopf bifurcation

In this section we investigate the Van der Pol system above the Hopf bifurcation. We fix $\epsilon = 0.1$. We introduce noise to the system to make the oscillations on the stable limit cycle irregular and fix $D = 0.1$, see Fig. 1. We apply time-delayed feedback to the system and look if and how the regularity and the timescales of the system can be controlled. The measure for the regularity of the oscillations is again the correlation time t_{cor} (3). We take the spectrum $S_{yy}(\omega)$ to characterize the timescales in the system.

4.1 Coherence of oscillations

We apply time-delayed feedback to the system like in the case below the Hopf bifurcation. In Fig. 7(a) we

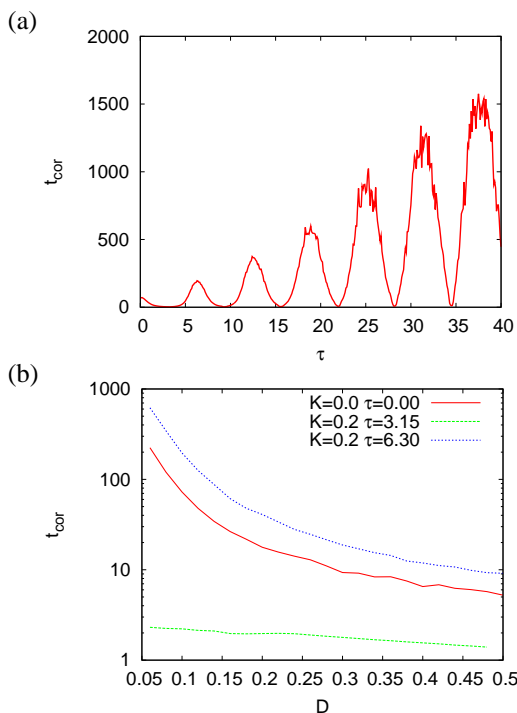


Figure 7. Correlation time t_{cor} of noisy oscillations in the Van der Pol system

(a) in dependence on delay time τ for $D = 0.1$, $K = 0.2$

(b) in dependence on noise intensity D

Parameters: $\omega_0 = 1$, $\epsilon = 0.1$

see that the coherence of oscillations can be controlled effectively by time-delayed feedback. The correlation time has maxima if τ is close to multiples of the basic period $T_0 = \frac{2\pi}{\omega_0}$ of the system and minima if τ is between those values. Time-delayed feedback can be used over a large range of noise intensities, Fig. 7(b), even above the Hopf bifurcation. We have the qualitatively same situation as below the Hopf bifurcation, Fig. 2, 4(a). The enhancement of t_{cor} for optimal values of τ is greater above the Hopf bifurcation. Below the Hopf bifurcation it was only

proportional to τ , see (26) and Fig. 4(a). We have also a different situation for the correlation time t_{cor} in dependence on the noise intensity D . Below the Hopf bifurcation t_{cor} was constant for $D \rightarrow 0$, Fig. 2. Above the Hopf bifurcation the correlation time goes to infinity for $D \rightarrow 0$, Fig. 7(b). The reason is the deterministic limit cycle in the system for $D = 0$. The dependence of t_{cor} on the feedback strength K for optimal values of τ is shown in Fig. 8. The correlation time t_{cor} is increasing monotonically with K as below the Hopf bifurcation, Fig. 5.

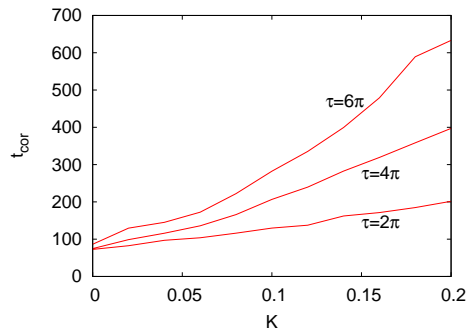


Figure 8. Correlation time t_{cor} of noisy oscillations in the Van der Pol system in dependence on feedback strength K for three different values of τ , $\omega_0 = 1$, $\epsilon = 0.1$, $D = 0.1$

In Fig. 9 we see the dependence of the second moment $\langle F^2 \rangle$ of the control force F on τ . We have minima for optimal values of τ like below the Hopf bifurcation, Fig. 4(b). Above the Hopf bifurcation we have the situation that the stable limit cycle vanishes for certain values of τ . These areas are shaded in Fig. 9. They are located at values of τ which are between the optimal values. In this case the control force F is approximately proportional to the amplitude of oscillations since $y(t - \tau)$ and $y(t)$ are on opposite sides of the origin and F is the difference of these values. Since the limit cycle breaks down, the amplitude of the oscillations gets smaller, and therefore $\langle F^2 \rangle$ becomes smaller in the shaded areas.

4.2 Timescales of oscillations

To characterize the timescales of the oscillations in the Van der Pol system we use again the spectrum $S_{yy}(\omega)$. In Fig. 10 we see the spectrum with applied time-delayed feedback. The control parameters τ and K are the same as below the Hopf bifurcation, Fig. 6(a). The spectra look very similar. We get additional peaks beside the main peak $\omega \approx \omega_0$. The peak frequencies (i.e. the frequency of oscillations) can also be controlled above the Hopf bifurcation, Fig. 11. We have the same piecewise linear dependence of the spectral peak period T_s on the delay time τ as below the Hopf bifurcation, Fig. 3(a). The frequency of the oscillations T_s can be changed within an interval around the basic period $T_0 = \frac{2\pi}{\omega_0}$.

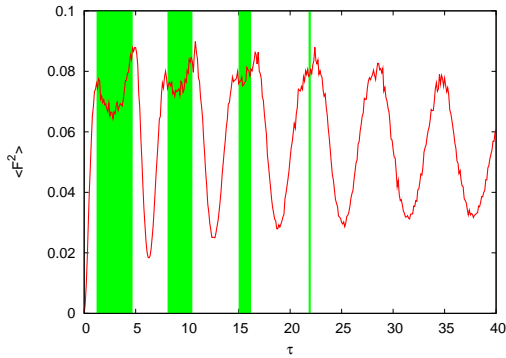


Figure 9. Second moment $\langle F^2 \rangle$ of the control force F from numerical simulations of the Van der Pol system in dependence on τ for $\epsilon = 0.1$, $D = 0.1$, $\omega_0 = 1$, $K = 0.2$, shaded area - limit cycle vanishes due to the control

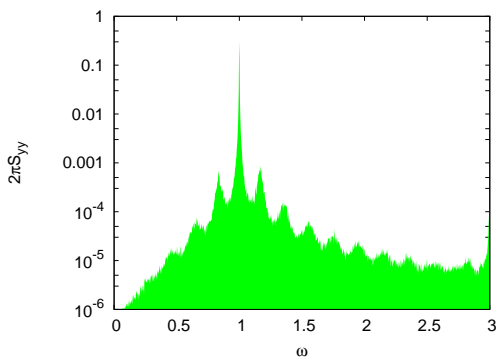


Figure 10. Numerically simulated spectrum $S_{yy}(\omega)$ of noisy oscillations in the Van der Pol system in the presence of delayed feedback for $\omega_0 = 1$, $\epsilon = -0.01$, $D = 0.1$, $\tau = 31.4$, $K = 0.2$

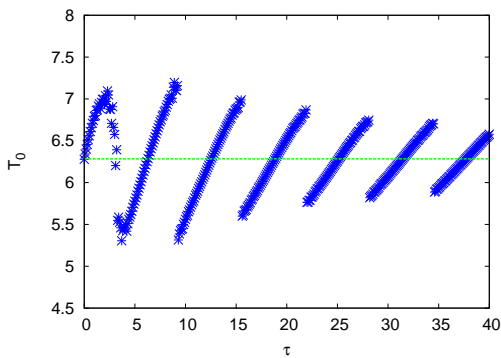


Figure 11. Spectrum peak periods T_s of noisy oscillations in the Van der Pol system in dependence on delay time τ for $\omega_0 = 1$, $\epsilon = 0.1$, $D = 0.1$, $K = 0.2$; solid line - basic period $T_0 = \frac{2\pi}{\omega_0}$

5 Conclusion

We have investigated the control of noisy oscillations of the paradigmatic nonlinear Van der Pol oscillator by a time-delayed feedback scheme. We have shown that time-delayed feedback can be used effectively below and above the Hopf bifurcation. By adjusting the control parameters τ and K we were able to modify important oscillation features like coherence and timescales. We have derived analytical results for the power spectral density and the correlation time for the noisy Van

der Pol system with applied control which are in excellent agreement with numerical simulations. Furthermore we have presented a mean field approximation of the uncontrolled Van der Pol system which takes into account the nonlinearity self-consistently and goes beyond the usual linearization. The mean field model describes the dependence of the correlation time upon noise intensity very well in a wide range of parameters. It provides a good approximation of the power spectral density even in the case of large noise and applied control.

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