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Control of noisy oscillations in the Van der Pol system

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0. Introduction

We present a mean field approximation for the noisy Van der Pol system with time-delayed feedback below the Hopf bifurcation. It goes beyond the usual linearization of a stable focus and takes into account the nonlinearity self-consistently.
We compare our analytical results to numerical simulations of the power spectral density and the correlation time in a regime with large noise intensity.

1. Model

Noisy Van der Pol oscillator with time-delayed feedback control.

$$\dot{x} = y$$
$$\dot{y} = -\omega_0^2 x + (\epsilon - x^2)\, y + \underbrace{D\,\xi(t)}_{noise} + \underbrace{K[\,y(t-\tau) - y(t)]}_{control} \qquad (1)$$

ϵ : Bifurcation parameter — ω_0 : Basic frequency

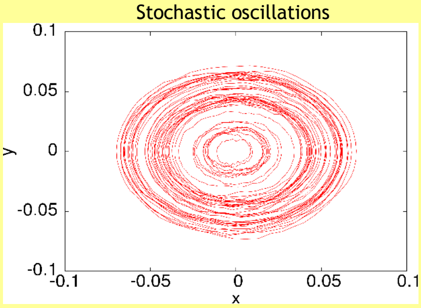
ξ : Gaussian white noise — D : Noise intensity

τ : Delay time — K : Feedback strength

2. Noise induced oscillations

Phase portrait of the system below the Hopf bifucation:

Stochastic oscillations



$\epsilon < 0$: Stable fixed point

$D = 0.003$
$\epsilon = -0.01$
$K = 0$
$\omega_0 = 1$

Aim: Use time-delayed feedback to control important oscillation features like **coherence** and **timescales [1]**.

3. Multivariate Ornstein-Uhlenbeck process

$$d\,\underline{x}(t) = -\underline{\underline{A}}\,\underline{x}(t)\,dt + \underline{\underline{B}}\,d\,\underline{W}(t)$$

$\underline{\underline{A}}$, $\underline{\underline{B}}$ $= const.$, $\underline{W}(t)$ *Wiener process*

Stationary variance matrix:

$$\underline{\underline{\sigma}} = \langle \underline{x}(t), \underline{x}^T(t)\rangle = \frac{(Det\,\underline{\underline{A}})\underline{\underline{B}}\,\underline{\underline{B}}^T + [\underline{\underline{A}} - (Tr\,\underline{\underline{A}})\underline{\underline{1}}]\,\underline{\underline{B}}\,\underline{\underline{B}}^T[\underline{\underline{A}} - (Tr\,\underline{\underline{A}})\underline{\underline{1}}]^T}{2(Tr\,\underline{\underline{A}})(Det\,\underline{\underline{A}})} \qquad (2)$$

Time correlation matrix:

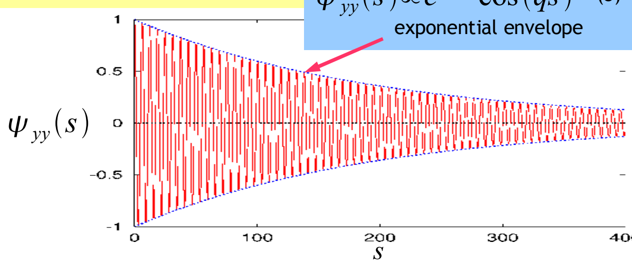
$$\psi_{x_j x_k}(s) = \langle x_j(t-s), x_k^T(t)\rangle = \underline{\underline{\sigma}}\,\underline{\underline{Q}}\exp\left[\begin{pmatrix}(-p+iq)s & 0\\ 0 & (-p-iq)s\end{pmatrix}\right]\underline{\underline{Q}}^{-1} \qquad (3)$$

eigenvalues of $\underline{\underline{A}}$: $p \pm iq$ (4)

Autocorrelation function:

$$\psi_{yy}(s) \propto e^{-sp}\cos(qs) \qquad (5)$$

exponential envelope



4. Mean field approximation of the Van der Pol system

Self-consistent linearization of the Van der Pol system without control (for $\epsilon<0$):

$$(\epsilon - x^2) \approx (\epsilon - \langle x^2\rangle) = \tilde{\epsilon} \qquad (6)$$

Van der Pol → Ornstein-Uhlenbeck

$$\underline{\underline{A}} = \begin{pmatrix}0 & -1\\ \omega_0^2 & -\tilde{\epsilon}\end{pmatrix} \quad \underline{\underline{B}} = \begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix} \qquad (7)$$

with (2) →

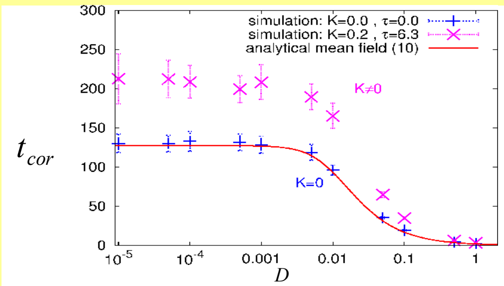
$$\langle x^2\rangle = \frac{D^2}{-2\tilde{\epsilon}\,\omega_0^2} \qquad (8)$$

self-consistent mean field approximation with (6),(8):

$$\tilde{\epsilon} = \frac{\epsilon}{2}\left(1 + \sqrt{1 + \frac{2D^2}{\epsilon^2\omega_0^2}}\right) \qquad (9)$$

5. Mean field approximation of the correlation time

$$t_{cor} = \frac{1}{\psi_{yy}(0)}\int_0^\infty |\psi_{yy}(s)|\,ds \overset{(5)}{\approx} \frac{2}{\pi}\frac{1}{p} \overset{(4),(7)}{=} \frac{2}{\pi}\frac{2}{|\tilde{\epsilon}|} \qquad (10)$$



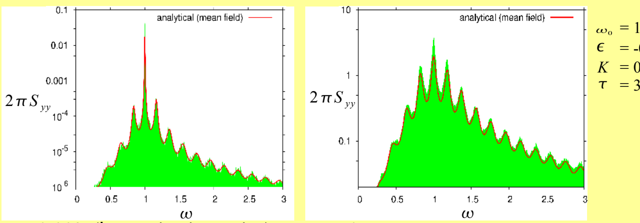
$\epsilon = -0.01$

Very good agreement between theory and simulations over a large range of noise intensities

6. Analytical approximation of the power spectral density

Power spectral density S_{yy} of linearized Van der Pol oscillator [2] with mean field $\tilde{\epsilon}$

$$S_{yy}(\omega) = \frac{D^2}{2\pi}\frac{\omega^2}{[\omega^2 - \omega_0^2 + \omega K\sin(\omega\tau)]^2 + \omega^2[\tilde{\epsilon} - K(1-\cos(\omega\tau))]^2} \qquad (11)$$



D=0.003 (low noise intensity) D=0.5 (large noise intensity)

Very good agreement between theory and simulations even for large noise intensities

7. Conclusions

- Van der Pol system with time-delayed feedback can be approximated by a mean field model.
- Very good agreement of the power spectral density and correlation time with simulations even for large noise intensities.

8. References

[1] N. B. Janson, A. G. Balanov, and E. Schöll. Delayed feedback as a means of control of noise-induced motion. *Phys. Rev. Lett.* 93,010601 (2004)
[2] E. Schöll, A. Balanov, N.B. Janson, and A. Neiman. Controlling stochastic oscillations close to a Hopf bifurcation by time-delayed feedback. *Stochastics and Dynamics* (2005)