TIME-DELAYED FEEDBACK CONTROL WITH VARIABLE PHASE-DEPENDENT COUPLING

P. Hövel and E. Schöll
Institut für Theoretische Physik, Technische Universität Berlin

Introduction

- Stabilization of unstable steady states by time-delayed feedback
- Analytic solution of characteristic equation by Lambert function
- Influence of phase-dependent coupling
- Domain of control for diagonal and nondiagonal control

Model and analytic results (diagonal coupling)

Generic case of an unstable focus with time-delayed control force (Pyragas control):

$$\frac{dx(t)}{dt} = \lambda x(t) + \omega y(t) - K[x(t) - x(t-\tau)]$$
$$\frac{dy(t)}{dt} = -\omega x(t) + \lambda y(t) - K[y(t) - y(t-\tau)]$$

K: Feedback gain, τ: Time delay, $\lambda, \omega \in R, \lambda > 0, \omega \neq 0$

Exponential ansatz: $x(t) \sim \exp(\Lambda t), y(t) \sim \exp(\Lambda t)$

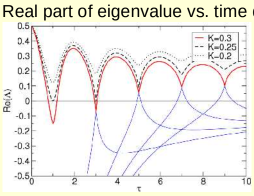
Characteristic equation: $\lambda \pm i\omega = \Lambda + K[1 - \exp(-\Lambda\tau)]$

Solution of characteristic equation by Lambert function W:

$$\Lambda\tau = W\left(K\tau e^{-(\lambda \pm i\omega)\tau + K\tau}\right) + (\lambda \pm i\omega)\tau - K\tau$$

Domain of control (diagonal coupling)

Real part of eigenvalue vs. time delay: $\lambda = 0.5, \omega = \pi$

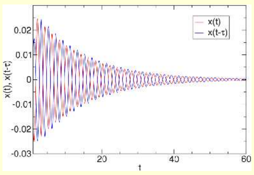


Start at uncontrolled eigenvalue $\lambda \pm i\omega$

Stabilization if $Re(\Lambda) < 0$

Infinite number of artificially created modes (see case $K = 0.3$)

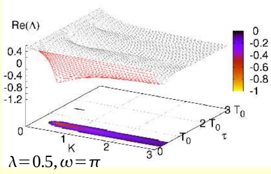
x-component of control force:



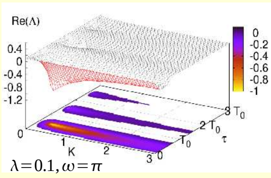
Current and delayed component in antiphase

Stabilization: Vanishing control force → Noninvasiveness

$Re(\Lambda)$ vs. time delay and feedback gain:



$\lambda = 0.5, \omega = \pi$



$\lambda = 0.1, \omega = \pi$

Bottom projection for stabilization ($\Lambda < 0$)

Absence of control ($K = 0$ or $\tau = 0$): $Re(\Lambda) = \lambda$

Analytic results of the domain of control:

- Minimum feedback gain:

$$K_{min} = \frac{\lambda}{2}$$

- No control for multiples of intrinsic period $T_0 = \frac{2\pi}{\omega}$

Variable phase-dependent coupling

Unstable focus with time-delayed control force and phase φ:

$$\frac{dx(t)}{dt} = \lambda x(t) + \omega y(t) - K\left(\cos(\varphi)[x(t) - x(t-\tau)] - \sin(\varphi)[y(t) - y(t-\tau)]\right)$$
$$\frac{dy(t)}{dt} = -\omega x(t) + \lambda y(t) - K\left(\sin(\varphi)[x(t) - x(t-\tau)] + \cos(\varphi)[y(t) - y(t-\tau)]\right)$$

$$\Leftrightarrow \begin{pmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{pmatrix} = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - K \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} x(t) - x(t-\tau) \\ y(t) - y(t-\tau) \end{pmatrix}$$

- Characteristic equation:

$$\lambda \pm i\omega = \Lambda + K e^{\pm i\varphi}\left(1 - e^{-\Lambda\tau}\right)$$

- Solution of characteristic equation by Lambert Function:

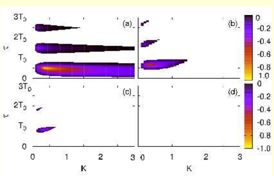
$$\Lambda\tau = W\left(K\tau e^{-(\lambda \pm i\omega)\tau + K[\cos(\varphi) \pm i\sin(\varphi)]\tau \pm i\varphi}\right) + (\lambda \pm i\omega)\tau - K\tau e^{\pm i\varphi}$$

- Minimum feedback gain:

$$K_{min}(\varphi) = \frac{\lambda}{\cos(\varphi) - \cos[\varphi + Im(\Lambda\tau)]} \geq \frac{\lambda}{2} = \left.\frac{\lambda}{\cos(\varphi) - \cos[\varphi + Im(\Lambda\tau)]}\right|_{\varphi = 0,\, Im(\Lambda\tau) = \pi}$$

Phase-dependent domain of control

Largest real part of eigenvalues vs. time-delay: $\lambda = 0.1, \omega = \pi, K = 0.3$

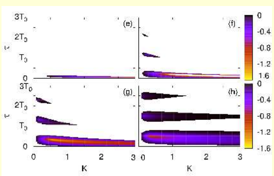


(a) $\varphi = 0$ (b) $\varphi = \pi/4$ (c) $\varphi = \pi/2$ (d) $\varphi = 3\pi/4$

For increasing phase:

- Smaller control regions
- Distortion to larger time delays
- Eventually no stabilization possible

→ For $\varphi = \pi$ no stabilization

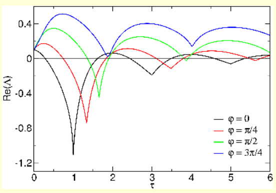


(e) $\varphi = 5\pi/4$ (f) $\varphi = 3\pi/2$ (g) $\varphi = 7\pi/4$ (h) $\varphi = 2\pi$

For phase approaching 2π:

- Larger control regions
- Distortion to smaller time delays

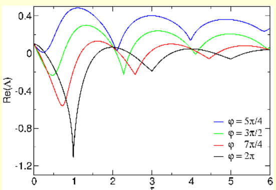
$Re(\Lambda)$ in dependence on time-delay and feedback gain for $K = 0.3$:



Start at uncontrolled eigenvalue

For increasing phase:

- Decrease of control interval [$Re(\Lambda) < 0$]
- Increase of maximum real part
- Shift to larger τ of minimum real part



For phase approaching 2π:

- Increase of control interval [$Re(\Lambda) < 0$]
- Decrease of maximum real part
- Shift to smaller τ of minimum real part

End at uncontrolled eigenvalue

→ Domain of control 2π periodic

Conclusion and Outlook

- 2π periodic modulation of domain of control for phase-dependent coupling
- Analytic results: Solution of characteristic equation by Lambert function, minimum feedback gain
- Application to optical systems (Lang-Kobayashi-mode of laser with feedback)

References

- P. Hövel and E. Schöll, Control of unstable steady states by time-delayed feedback methods, Physical Review E **72**, 046203 (2005)
- A. Amann, E. Schöll, and W. Just, Some basic remarks on eigenmode expansions of time-delay dynamics, Physica A (2005), in print