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Mechanical pressure and work cycle of confined active Brownian particles

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Abstract – We derive an analytic expression for the mechanical pressure of a generic one-dimensional model of confined active Brownian particles (ABPs) that is valid for all values of Péclet number Pe and all confining scenarios. Our model reproduces the known scaling of bulk pressure with Pe^2 while in strong confinement pressure scales with Pe . Our analytic results are very well reproduced by simulations of ABPs in 2D. We use the pressure formula to calculate both the work performed by an active engine and its efficiency. In particular, efficiency is maximized for work cycles with finite period and not in the limit of infinitely slow cycles as in thermodynamic engines.

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Introduction. – The properties and structure formation of active systems are quite different as compared to their equilibrium counterparts [1–7]. This becomes particularly apparent in confinement [8–17]. Indeed, active particles accumulate at walls [18], interfaces [19,20], as well as obstacles [21,22], and in denser suspensions they show motility-induced phase separation [23]. One of the key macroscopic quantities of interest is the mechanical pressure Π that active particles exert on confining walls [10,24–26]. Such a quantity is crucial for determining the performance of devices rectifying active motion [27–29], for work cycles that exploit active baths [30], for invasion of active particles into confining space [31], as well as evaporation [32] and wetting [33,34] in active fluids.

Predicting the value of Π for active systems is not trivial since, due to the active nature of the particles, pressure is, in general, no longer a thermodynamic state function [35]. It is not even an intensive variable since it depends explicitly on extensive variables (like number of particles), as we show below. Several articles [10,24,25] have reported independently the characteristic scaling for the pressure, $\Pi \propto Pe^2$, where the Péclet number $Pe = v_{\text{act}}R/D$ depends on the active velocity v_{act} , the linear size R , and

the diffusion coefficient D of the particle. This relation for pressure has been derived for semi-infinite systems. However, the dynamics of active Brownian particles (ABPs) is very sensitive to the presence of boundaries [8–16] and it is not obvious that the scaling for the pressure also holds for *confined* ABPs. Indeed, recent numerical works [9,10] have shown that the pressure of strongly confined active particles scales as $\Pi \propto Pe$. Therefore, the scaling of Π with Pe depends on the system size. At the moment, a comprehensive relation, valid for all confining scenarios, between the pressure and the microscopic parameters (such as active velocity and tumbling rate), which control the dynamics of ABPs, is still lacking.

In this letter we derive a closed-form expression for the pressure exerted by confined ABPs that is valid for all values of Pe and all confining scenarios. In order to do so, we consider simple ABPs that only move in one dimension either along the x -axis (“up” state) or against it (“down” state) and that tumble between both states. Furthermore, the ABPs experience a confining soft potential such that, in the limit of diverging potential strength, our model retrieves the case of ABPs confined in a box with hard walls. In this standard case, the calculated mechanical pressure displays multiple scalings with Pe . In particular, when particles undergo multiple tumbling events between subsequent collisions with the walls (diffusive regime), pressure

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scales as $\Pi \propto \text{Pe}^2$ in agreement with refs. [10,24,35]. In contrast, for strongly confined active colloids or for very large values of Pe (as the one attained for dry macroscopic active matter [36–41]) particles only undergo a few (if at all) tumbling events (ballistic regime) and the pressure scales as $\Pi \propto \text{Pe}$. Our analytic predictions for the pressure are in very good agreement with results from numerical simulations of ABPs in two dimensions, with the numerical results of refs. [9,10] and with the expansion approach of ref. [34] which exclusively treats the case where pressure scales as $\Pi \sim \text{Pe}^2$. Thus, despite its simplicity our model captures the essence of the dynamics of confined ABPs for all values of Pe . Hence, it can be used to predict the mechanical pressure of ABPs on both the microscopic ($\sim \mu\text{m}$) and macroscopic ($\sim \text{cm}, \text{m}$) scale in all confining scenarios.

We apply the analytical formula for the mechanical pressure to the recently introduced work cycle of active engines [30,42–44]; devices that exploit the capability of a bath of ABPs to perform directed work (see also a recent review on the topic ref. [45]). We find that the work is governed by two dimensionless parameters and that the efficiency of quasistatic work cycles is optimal for a finite period in contrast to thermodynamic engines.

Model. – The N noninteracting ABPs experience the confining potential

$$\beta U(z) = \begin{cases} f(z/L - 1), & z > L, \\ 0, & -L \leq z \leq L, \\ -f(z/L + 1), & z < -L, \end{cases} \quad (1)$$

where L is the size of the system (not including the soft walls), f controls the softness of the walls, $\beta = 1/(k_B T)$, k_B is the Boltzmann constant, and T is the temperature. Within the overdamped regime, the time evolution of the reduced densities for up and down states, “up” (ρ_\uparrow) and “down” (ρ_\downarrow) states, which we express as functions of the dimensionless position $x = z/L$ and with time in units of L^2/D , are governed by

$$\dot{\rho}_\uparrow(x) = -\partial_x J_\uparrow - \Gamma \frac{L^2}{R^2} [\rho_\uparrow(x) - \rho_\downarrow(x)], \quad (2a)$$

$$\dot{\rho}_\downarrow(x) = -\partial_x J_\downarrow + \Gamma \frac{L^2}{R^2} [\rho_\uparrow(x) - \rho_\downarrow(x)]. \quad (2b)$$

(For simplicity we do not denote explicitly the dependence on time.) In eqs. (2) we have identified the fluxes as

$$J_\uparrow(x) = - \left[\partial_x \rho_\uparrow(x) - \frac{L}{R} \text{Pe} \rho_\uparrow(x) + \rho_\uparrow(x) \beta \partial_x U(x) \right], \quad (3a)$$

$$J_\downarrow(x) = - \left[\partial_x \rho_\downarrow(x) + \frac{L}{R} \text{Pe} \rho_\downarrow(x) + \rho_\downarrow(x) \beta \partial_x U(x) \right], \quad (3b)$$

and we have introduced

$$\text{Pe} = v_{\text{act}} R/D, \quad \Gamma = \gamma R^2/D, \quad (4)$$

the particle Péclet number Pe and dimensionless tumbling rate Γ defined as tumbling rate γ times the diffusion time scale R^2/D . For later use, we note that $\text{Pe}^2/\Gamma = v_{\text{act}}^2 \gamma^{-1}/D$ is the ratio of active to passive diffusion coefficients. We note the ratio $\text{Pe}/\Gamma = v_{\text{act}}/(\gamma R)$ can be identified with the dimensionless rotational Péclet number Pe_r .

By solving eqs. (2) in the steady state using piecewise solutions in the three regions of $\beta U(x)$ (see sect. S1 of the Supplementary Material **Supplementary material.pdf** (SM)), we compute the dimensionless mechanical pressure exerted on the right wall (the same results hold for the left wall),

$$\Pi = \int_1^\infty [\rho_\uparrow(x) + \rho_\downarrow(x)] f dx. \quad (5)$$

Pressure. – In order to study the case of ABPs confined within a box, we take the limit of the hard-core potential ($f \rightarrow \infty$, see sect. S1 of the SM). In this limit, Π becomes

$$\Pi_\infty = \bar{\rho} \frac{R^2}{L^2} \frac{\kappa_c^3 \cosh \kappa_c}{\text{Pe}^2 \sinh \kappa_c + 2\Gamma \kappa_c \cosh \kappa_c}. \quad (6)$$

Here, $\bar{\rho} = NR/2L$ is the dimensionless number density and

$$\kappa_c = \kappa L = \frac{\sqrt{\text{Pe}^2 + 2\Gamma}}{R} L, \quad (7)$$

where κ is the inverse of the effective length that characterizes the exponential decay of the density profile close to the wall¹. We remark that κ depends solely on microscopic parameters and not on the system size. In particular, when $\kappa_c \gg 1$, eq. (6) is approximated by

$$\Pi_\infty \simeq \bar{\rho} \frac{R^2}{L^2} \frac{\kappa_c^3}{\text{Pe}^2 + 2\Gamma \kappa_c}. \quad (8)$$

The regime $\kappa_c \gg 1$ is typical for active matter as it occurs whenever either $\text{Pe} \gg R/L$ or $\Gamma \gg R^2/L^2$ (see footnote ²). The latter means that during passive diffusion across the system, tumbling occurs frequently. Therefore, in the following we focus on the relevant case of $\kappa_c \gg 1$. In sect. S2 of the SM we present the mathematical derivation of all limiting regimes discussed in this letter and argue that the results are valid even for moderately large values of κ_c .

Figures 1(a) and (b) show that Π_∞ grows monotonically upon increasing Pe , as expected. In the limit of small Péclet numbers, $\text{Pe} \ll \text{Pe}_{\text{small}} = \sqrt{2\Gamma}$, eq. (8) gives

$$\Pi_\infty^{\text{small}} \simeq \bar{\rho} \left(1 + \frac{\text{Pe}^2}{2\Gamma} \right), \quad \text{Pe} \ll \text{Pe}_{\text{small}}. \quad (9)$$

Thus, for vanishingly small Pe , Π_∞ reduces to its equilibrium value $\Pi_\infty^0 = \bar{\rho}$. We note that the expression in

¹The exponential decay and the associated decay length (eq. (7)) are valid for all values of Pe and L . In particular, for small values of Pe eq. (7) reduces to $\kappa \simeq \Gamma$, in agreement with ref. [18].

²We remark that if $L/R > 10$, $\kappa_c \gg 1$ provided that $\text{Pe} \geq 0.1$ or $\Gamma \geq 0.01$.

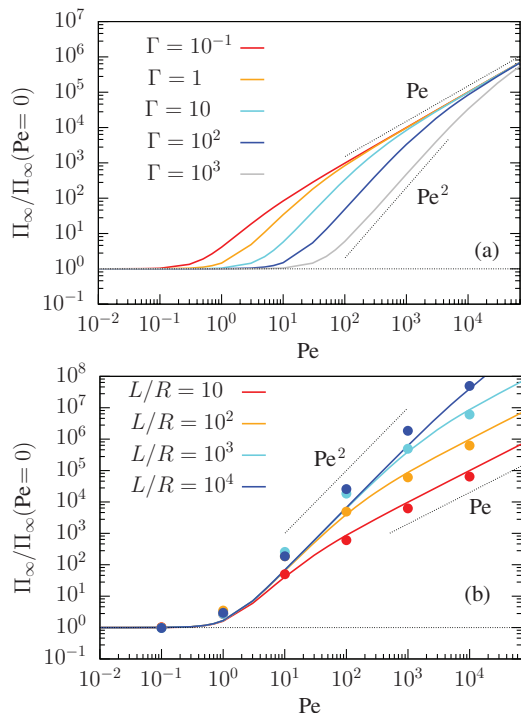


Fig. 1: (a) Pressure Π_∞ as a function of Pe for diverse values of Γ and $L = 10R$. (b) Π_∞ as a function of Pe for $\Gamma = 3/4$ and diverse values of L/R . The dots present the results of 2D numerical simulations of ABPs characterized by the same value of Pe , the linear size L and Γ , and with no fitting parameters (see sect. S6 of the SM).

the brackets agrees with the effective temperature $T_{\text{eff}} = T(1 + Pe^2/2\Gamma)$ introduced, for example, in refs. [46,47] (see also sect. S3 of the SM). We remark that Π_∞^0 does not depend explicitly on the system size L or the particle number N , as required for an intensive thermodynamic quantity. For $Pe \simeq Pe_{\text{small}}$, activity starts to dominate the pressure. Using eqs. (4), the condition $Pe \simeq Pe_{\text{small}}$ implies $D_{\text{act}} = v_{\text{act}}^2/\gamma \simeq D$. Thus, the pressure starts to grow with Pe^2 when the active contribution to the total diffusion coefficient $D_{\text{eff}} = D + D_{\text{act}}$ becomes dominant [46,47].

For large Péclet numbers ($Pe \gg Pe_{\text{small}}$) we obtain from eq. (8):

$$\Pi_\infty^{\text{large}} \simeq \bar{\rho} \frac{Pe^2}{2\Gamma}, \quad \text{for } Pe_{\text{large}} \gg Pe \gg Pe_{\text{small}}, \quad (10a)$$

$$\Pi_\infty^{\text{large}} \simeq \bar{\rho} \frac{L}{R} Pe, \quad \text{for } Pe \gg Pe_{\text{large}}, Pe_{\text{small}}, \quad (10b)$$

where $Pe_{\text{large}} = 2\Gamma L/R$. When $Pe \gg Pe_{\text{large}}$, pressure Π_∞ attains its asymptotic form (10b), $\Pi_\infty \simeq N Pe/2$, growing linearly in Pe . In this regime the pressure is no longer an intensive variable since it depends explicitly on the particle number N .

The crossover between the different scalings of Π_∞ with Pe occurs at $Pe = Pe_{\text{large}} = 2\Gamma L/R$, *i.e.*, when the ballistic time L/v_{act} is comparable to the mean run time $1/\gamma$ between two tumbling events. Thus, in the asymptotic

regime where $L/v_{\text{act}} \ll 1/\gamma$, particles only undergo a few (if at all) tumbling events between subsequent collisions with the walls (ballistic regime). As a result, they spend the large amount of time at the boundaries. Therefore, the pressure increases linearly with Pe and depends explicitly only on the particle number and not on the system size. In the opposite case $L/v_{\text{act}} \gg 1/\gamma$ (*i.e.*, $Pe \ll Pe_{\text{large}}$), the particles undergo multiple tumbling events between subsequent collisions with the walls (diffusive regime). In this regime, increasing Pe has a twofold effect: first, it directly enhances pressure when the particles hit the wall, and second, it reduces the number of tumbling events between two subsequent collisions at the walls and thereby enhances the density of the particles at the wall. This twofold effect explains the quadratic dependence of Π_∞ on Pe in eq. (10a), as shown in both panels of fig. 1.

Accordingly, fig. 1(a) shows that the scaling of the pressure with Pe changes dramatically upon changing the tumbling rate Γ . At the micrometric scale, this result is crucial for confined bacterial suspensions [48,49], whose tumbling rate depends on both the biology of the bacteria as well as on external control parameters³. At the macroscopic scale our result is crucial for determining the pressure of dry active matter, such as small robots [37], ants [38,39], sheep [40], and humans [36,41], just to mention a few among others.

For active colloids, such as Janus particles, Γ is controlled by the rotational diffusion coefficient D_{rot} , which depends on their size and shape. For spherical particles $D_{\text{rot}} = \frac{3}{4} \frac{D}{R^2}$, which in 2D equals γ so that $\Gamma = 3/4$. For this case, fig. 1(b) presents Π_∞ *vs.* Pe for diverse system sizes. Interestingly, for typical values of the Péclet number ($Pe \simeq 1-100$) and system sizes ($L \gtrsim 100R$) that have been investigated experimentally [50] or numerically [51], our model predicts $\Pi_\infty \propto Pe^2$, in agreement with refs. [10,24,25,35,51,52]. However, for smaller system sizes $L \simeq 10R$ or for very large values of the Péclet number, $Pe \gg Pe_{\text{large}}$, the asymptotic behavior $\Pi_\infty \propto Pe$ is retrieved.

In order to check the validity of our expression against more realistic models, we performed 2D simulations of spherical ABPs characterized by $\Gamma = 3/4$, where the particle orientations diffused on the unit circle (see sect. S6 of the SM for more details on the simulations). The results are included in fig. 1(b). Interestingly, without using any fitting parameters, the agreement between the theoretical predictions and the results of the numerical simulations is very good for all values of Pe and L that we tested. Hence, our simple two-state model captures the essence of the dynamics of confined ABPs [53].

Compressibility. – Having an explicit expression for the pressure (cf. eq. (6)), we can calculate explicitly the dimensionless compressibility $H_\infty = -(\partial\Pi_\infty/\partial L)^{-1}/L$ (see

³Typically, $\Gamma \gtrsim 100$ [48]. Even upon genetically switching off tumbling, the lower bound is $\Gamma \simeq 1$ due to rotational diffusion. In addition, in chemical gradients Γ can vary by a factor of five [49].

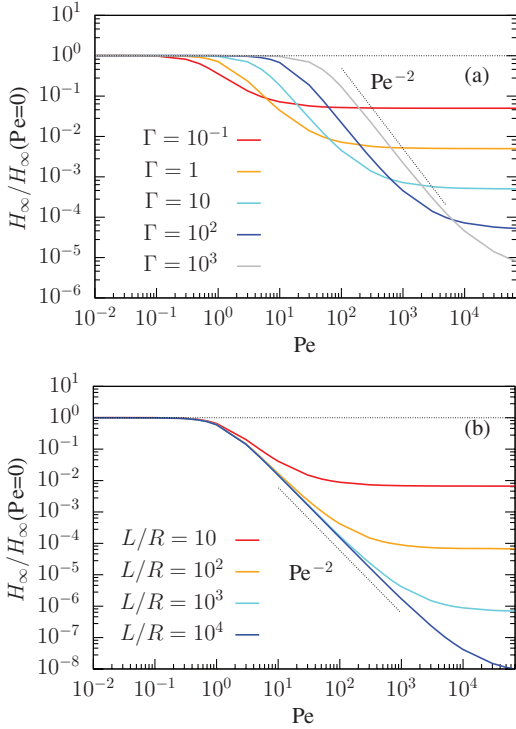


Fig. 2: (a) Compressibility H_∞ as a function of Pe for diverse values of Γ and for $L/R = 10$. (b) H_∞ as a function of Pe for diverse values of L/R and for $\Gamma = 3/4$.

sect. S5 of the SM). For the relevant case of $\kappa_c \gg 1$, it reduces to

$$H_\infty \simeq \frac{1}{\Pi_\infty} \left(1 + \frac{Pe^2}{2\Gamma\kappa_c} \right). \quad (11)$$

Figure 2(a) shows H_∞ plotted *vs.* Pe for $L = 10R$ and diverse values of Γ . Similar to the behavior of pressure, upon increasing Pe beyond $Pe_{\text{small}} = \sqrt{2\Gamma}$, the compressibility starts to decrease as Pe^{-2} due to the prefactor $1/\Pi_\infty$ (cf. eq. (9)) and then, beyond $Pe \simeq Pe_{\text{large}} = 2\Gamma L/R$, it reaches the normalized plateau value $(N\Gamma L/R)^{-1}$, *i.e.*, the compressibility depends on extensive variables such as N and L . This is in contrast to equilibrium systems, for which compressibility is an intensive variable whose corrections due to finite size become negligibly small upon increasing system size. In particular, larger values of Γ delay the onset of the decrease of H_∞ and they also lower the plateau value since more tumbling releases pressure generated by the active particles. Figure 2(b) shows the relevant case of active colloids and that not only pressure Π_∞ but also H_∞ retains a dependence on the system size.

Active engine. – We can exploit the exact expression for the pressure to calculate the work performed by the system during the periodic work cycle shown in fig. 3(a). Work is performed by the system only when there is a change in the system size $2L$. Assuming that these changes are sufficiently slow so that the mechanical pressure can adjust instantaneously, the total dimensionless work per

particle along a cycle reads:

$$W = \int_{L_1}^{L_2} \frac{\Pi_\infty(L, Pe_2)}{N} \frac{2dL}{R} - \int_{L_1}^{L_2} \frac{\Pi_\infty(L, Pe_1)}{N} \frac{2dL}{R}. \quad (12)$$

Figure 3(b) shows that for small system sizes the work performed by the system grows linearly with L_1 (for fixed ratio L_2/L_1). In this regime the work is insensitive to Γ (blue and cyan curves as well as red and orange curves lie on top of each other in fig. 3(b)), while the overall amount of work depends on Pe . Upon increasing L_1 further, the work W reaches a plateau. Here, W increases upon decreasing Γ for both values of Pe , since tumbling reduces the pressure of the expanding system. The dependence of W on Γ shows that the work performed by the active system explicitly depends on the tumbling rate, *i.e.*, on some microscopic time scale. Such a dependence does not occur in passive systems and therefore is a signature of the active nature of the system under study. In contrast, the dependence of W on Pe is clear since the active motion of the ABPs generates the force with which they push against the wall.

All these observations can be rationalized by considering the limit $\kappa_c \gg 1$ in eq. (12) which gives

$$W \simeq W_0 \ln \left[\frac{1 + L_2/\Lambda}{1 + L_1/\Lambda} \right] - \ln \left[\frac{L_2}{L_1} \right]. \quad (13)$$

Here, we have identified the effective length Λ and strength W_0 of the work cycle,

$$\Lambda = \frac{Pe^2 R}{2\Gamma \sqrt{Pe^2 + 2\Gamma}}, \quad W_0 = 1 + \frac{Pe^2}{2\Gamma}. \quad (14)$$

After rescaling work by W_0 and system size by Λ , for $W_0 \gg 1$ all curves from fig. 3(b) collapse onto one master curve, as demonstrated in fig. 3(c). In the regime $Pe \gg Pe_{\text{small}} = \sqrt{2\Gamma}$ the parameters become

$$\Lambda \simeq \frac{v_{\text{act}}}{\gamma}, \quad W_0 \simeq \frac{v^2}{D\gamma} = \frac{D_{\text{act}}}{D}, \quad (15)$$

i.e., the threshold length Λ is the typical distance traveled by the particle between two tumbling events, whereas the work strength W_0 is proportional to the ratio of active to passive diffusion coefficients. In particular, for large systems sizes, $L_1, L_2 \gg \Lambda$, we have

$$W \approx (W_0 - 1) \ln(L_2/L_1) \simeq W_0 \ln(L_2/L_1). \quad (16)$$

Thus, the work per particle over one cycle solely depends on W_0 and its dependence on system size is reminiscent of the work done by a passive ideal gas.

Efficiency. – Finally, we define the efficiency of the work cycle as the ratio of the total work performed by the system composed of N particles,

$$W^{\text{tot}} = NW, \quad (17)$$

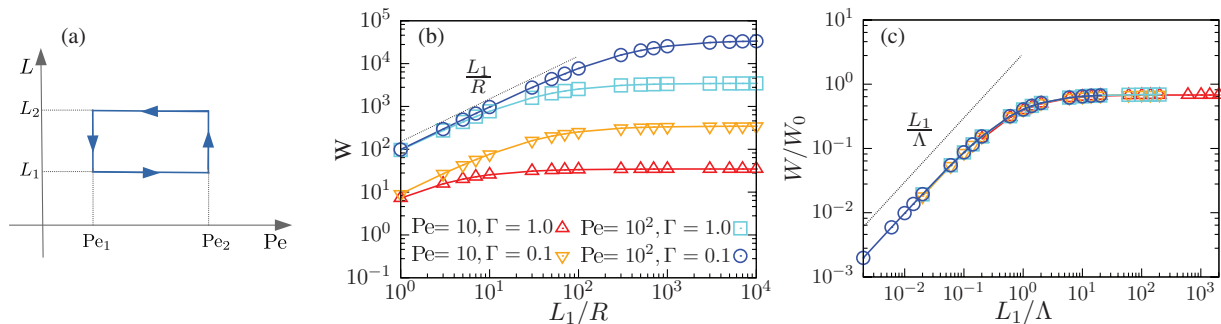


Fig. 3: (a) Scheme of the work cycle of the active engine in the Pe - L plane. (b) Work W performed by one cycle of the active engine as a function of L_1 with $L_2 = 2L_1$ and $Pe_1 = 0$ for diverse values of $Pe_2 = Pe$ and Γ . (c) Rescaled data of panel (b).

to the total energy injected into the system,

$$\eta = W^{\text{tot}} / (W^{\text{tot}} + W_{\text{irr}}), \quad (18)$$

where W_{irr} accounts for the energy spent in dissipative processes. In the following we assume that the change in the value of Péclet number occurs with no additional dissipation, as it happens for example for light-controlled active colloids whose activity can be tuned by shading the light [54,55]. Accordingly, W_{irr} is the sum of two contributions. First, we express the power dissipated by the particles due to their active motion as $W_{\text{irr}}^{\text{pcl}} = N\mathcal{P}\tau$, where \mathcal{P} is the mean power⁴ dissipated by a single particle during one work cycle of period τ . We remark that for infinitely slow processes $\tau \rightarrow \infty$, $W_{\text{irr}}^{\text{pcl}} \rightarrow \infty$ and hence $\eta \rightarrow 0$. Therefore, active engines should be run at finite-time compression and expansion, a regime in which the additional dissipation due to friction forces acting when the container expands or compresses, $W_{\text{irr}}^{\text{sys}}$, should be accounted for [56]. Very generally, the power dissipated by friction forces can be expressed as

$$\dot{W} = F \cdot v, \quad (19)$$

where F is the applied force and v is the velocity. Within linear response theory [57] we have $F = \mathcal{W}v$, where \mathcal{W} is the friction coefficient. Thus, the dissipated power reads

$$\dot{W} = \mathcal{W}v^2. \quad (20)$$

In the case under study, we estimate the velocity via the change in the volume $v \simeq 2\Delta L/\tau$ during the cycling time τ . Accordingly, the dissipated power is

$$\dot{W} = 4\mathcal{W}\frac{\Delta L^2}{\tau^2}, \quad (21)$$

and hence the dissipated work during the cycle reads

$$W_{\text{irr}}^{\text{sys}} = \dot{W}\tau = 4\mathcal{W}\frac{\Delta L^2}{\tau}. \quad (22)$$

We remark that, at leading order, $W_{\text{irr}}^{\text{sys}}$ does not depend on the parameters characterizing the active particles, in

⁴Note that \mathcal{P} is an average over free particles and particles trapped at the wall.

particular, it does not depend on N . Assuming instantaneous changes of Pe , the velocity is $2(L_2 - L_1)/\tau = 2\Delta L/\tau$ and $W_{\text{irr}}^{\text{sys}} = 4\mathcal{W}\Delta L^2/\tau$ [57,58], where \mathcal{W} plays the role of an effective friction coefficient. All in all we obtain

$$W_{\text{irr}} = W_{\text{irr}}^{\text{pcl}} + W_{\text{irr}}^{\text{sys}} = N\mathcal{P}\tau + 4\mathcal{W}\Delta L^2/\tau, \quad (23)$$

where \mathcal{P} and \mathcal{W} are phenomenological parameters encoding, respectively, for the power dissipated by the active particles⁵, which depends on the specific propulsion mechanism, and for the effective friction of the container. Maximizing the efficiency with respect to τ amounts to minimizing W_{irr} which gives an optimal time of the cycle

$$\tau_{\text{opt}} = (4\mathcal{W}\Delta L^2/N\mathcal{P})^{1/2}. \quad (24)$$

Accordingly, even for quasi-static expansions, the efficiency η is maximized for a *finite* cycle time τ in stark contrast to the quasi-static limit $\tau \rightarrow \infty$ of thermodynamic engines for which $W_{\text{irr}}^{\text{pcl}}$ is replaced by the heat Q that is independent of τ . This is in agreement with the numerical results of ref. [30]. Interestingly, τ_{opt} depends on the ratio between the dissipation in the system, $4\mathcal{W}\Delta L^2$ and that due to the active bath $N\mathcal{P}$. Hence, active engines exploiting many particles should be run at short cycling times, whereas the opposite holds for smaller particle numbers.

In the above calculations we have assumed that the particle density follows adiabatically the change in volume ΔL . We recall that the relaxation time of the system can be estimated as

$$\tau_{\text{relax}} \simeq \min\left(\frac{\Delta L^2}{D}, \frac{\Delta L}{v_{\text{act}}}\right). \quad (25)$$

⁵For bacteria, \mathcal{P} amounts to the metabolic cost of keeping the bacteria alive and swimming. For diffusiophoretic colloids, \mathcal{P} amounts to the power dissipated in order to keep the imbalance in the bulk chemical potentials of the reactants and the reaction products. For light-driven phoretic colloids, \mathcal{P} amounts to the power dissipated (per particle) by the light source. For ABPs, under the assumption that all the energy “consumed” by the internal mechanism responsible for active displacement is transformed into motion, we have $\mathcal{P}_{\text{ABP}} = F \cdot v_0 = \frac{k_B T}{D} v_0^2$, where F is the effective driving force.

Hence, the adiabatic assumption we made in deriving eq. (12) is fulfilled when $\tau_{\text{relax}} \ll \tau_{\text{opt}}$. For $\tau_{\text{relax}} = \Delta L/v_{\text{act}}$ that leads to

$$N \ll \frac{4Wv_{\text{act}}^2}{\mathcal{P}}. \quad (26)$$

Interestingly, eq. (26) shows that the maximum efficiency can be attained by small systems, whereas larger systems containing a larger number of particles N , will be suboptimal.

Conclusions. – Based on a one-dimensional model for run-and-tumble particles, we have derived an analytic expression for the mechanical pressure ABPs exert on bounding walls. In the limit of large systems we reproduce the well-known scaling of the bulk pressure with Pe^2 . In contrast, for either strongly confined micrometric ABPs or macroscopic ABPs with very large Pe , the pressure scales with Pe and is no longer an intensive variable. We clearly rationalize the regimes where the different scalings are observed. Furthermore, two-dimensional Brownian dynamics simulations of ABPs quantitatively agree with our analytic expression and thereby show its generality.

Our analytic formula for pressure allows to systematically explore basic features of confined active systems between bulk- and surface-driven behavior. For the recently introduced active engines we have calculated the work performed during one cycle in the quasi-static limit. It explicitly depends on the characteristic time scale Γ^{-1} , a feature that is absent in conventional thermodynamic engines. Furthermore, the efficiency is maximized at a *finite* cycle rate due to the inherent dissipation, in clear contrast to thermodynamic engines where infinitely small rates avoid dissipation. Surprisingly, such an optimal cycling time is typical of “small engines”, *i.e.*, those engines exploiting a small number of active particles.

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