17.5.10

5. Übung nichtlineare Dynamik & Kontrolle

Bemerkungen zum 2. Zettel:

Aufgabe 4.1:

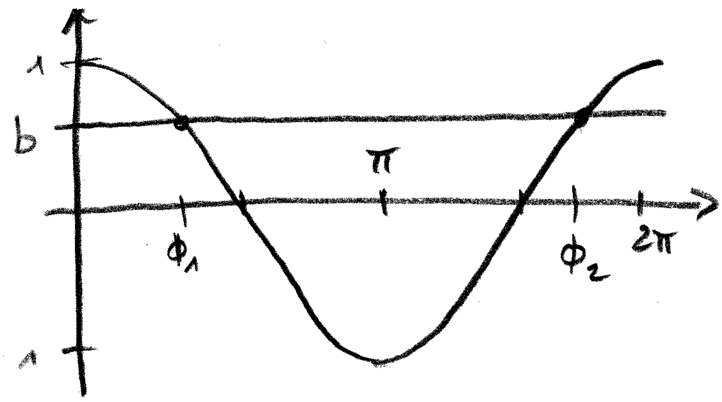
Fixpunkte:

$$\dot{r} = r(1-r^2)$$
$$\dot{\phi} = b - r\cos\phi$$

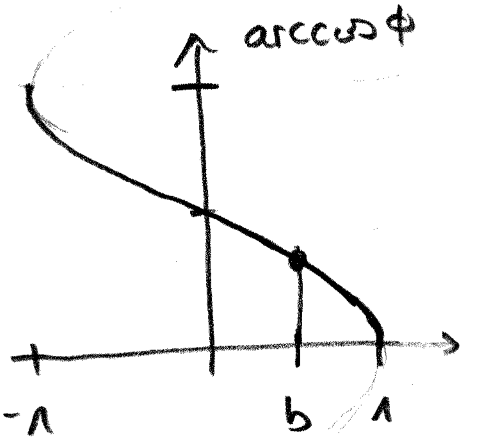
Fixpunkte auf dem Einheitskreis $r=1$:

$$0 = b - \cos\phi$$

Zwei FP in $[0, 2\pi)$



Aber



Gibt nur eine Lsg.

$$\phi = \arccos b$$

Andere Lsg

$$\phi = -\arccos b$$

oder wenn $\phi \in [0, 2\pi)$ sein soll

$$\phi = 2\pi - \arccos b$$

Diese Übung: Detaillierte Besprechung des Papers (siehe hinten)

E. Ott, C. Grebogi, J. A. Yorke,
"Controlling Chaos", Phys. Rev. Lett. 64, 1196 (1990)
über 3000 mal zitiert.

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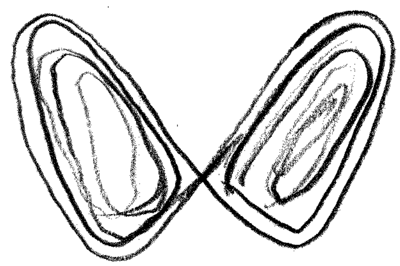
① Motivation:

- Chaotisches System soll kontrolliert werden
- Ein zugänglicher Systemparameter wird zeitlich um einen kleinen Wert verändert → Stabilisierung eines periodischen Orbits

② Was soll stabilisiert werden?

- Chaotischer Attraktor enthält ∞ viele periodische Orbits.

Bsp: Lorenz-Attraktor



- Ziel: einen instabilen periodischen Orbit zu stabilisieren
- System wird nur wenig verändert

3 Annahme:

- Zur Vereinfachung nur 3D-Systeme (geht aber auch in höheren Dimensionen)

$$\frac{d}{dt}\underline{x} = \underline{F}(\underline{x}, p) \qquad x \in \mathbb{R}^3$$

p - Systemparameter der verändert werden kann

4 Anwendung auf Experimente:

- Methode Funktioniert auch wenn genaue Dynamik nicht bekannt ist
- Wenn nur eine Variable gemessen werden kann, kann man eine Delay-Embedding verwenden:

$$\underline{X}(t) = \big(z(t),\ z(t-T),\ \dots,\ z(t-MT)\big)$$

gemessene Variable

Vektor spannt für geeignete T den Phasenraum auf

⑤ System Parameter:

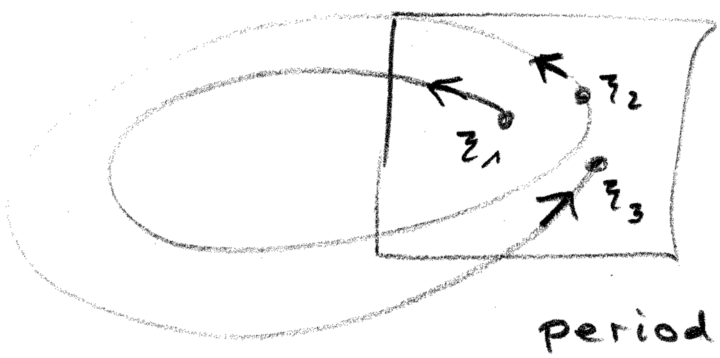
Annahme:

- p kann in einem kleinen Bereich um p_0 variiert werden
- O.B.d.A: $p_0 = 0$

$\rightarrow \quad -p_* < p < p_*$

⑥ Poincare Ebene:

- Betrachte nur Poincare Schnitte



Vereinfachung

3D Fluss

$\rightarrow$ 2D Map

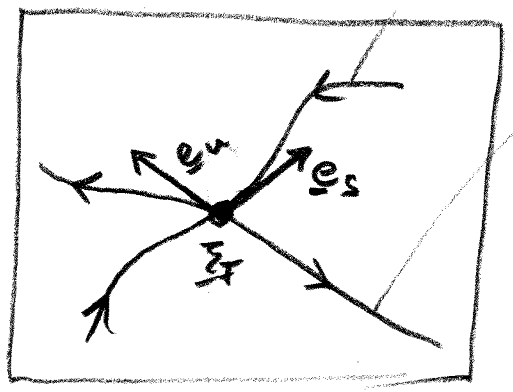
periodischer Orbit periode 1

$\rightarrow$ Fixpunkt

7a Benötigtes Wissen über Fixpunkt:

- Sei ξ_F der zu stabilisierende FP (O.B.d.A $\xi_F = 0$)
- FP muss ein Sattel sein

stabile Mannigf.



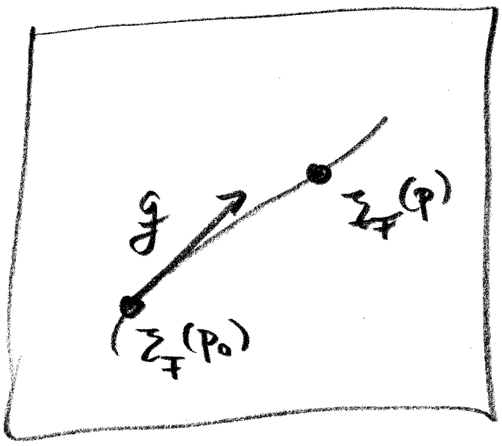
instabile Mannigf.

Wir benötigen

e_u, e_s, λ_u, λ_s

Eigenwerte

(7b) **Wie verschiebt sich der FP wenn p verändert wird?**



$$\underline{g} := \left.\frac{\partial \underline{\xi}_F(p)}{\partial p}\right|_{p_0}$$

(8) **Duale Vektoren:**

- Die Vektoren $\underline{e}_u$ und $\underline{e}_s$ sind nicht orthogonal.
- Betrachte einen Vektor $\underline{v}$. welche Koeffizienten hat $\underline{v}$ bzgl. $\underline{e}_u$ und $\underline{e}_s$?

$$\rightarrow \quad \underline{v} = \alpha\, \underline{e}_u + \beta\, \underline{e}_s$$

$$\rightarrow \quad \alpha = ?$$

- verwende Duale Basis $\underline{f}_u, \underline{f}_s$ mit

$$\underline{f}_u \cdot \underline{e}_u = 1 \qquad \underline{f}_u\, \underline{e}_s = 0$$
$$\underline{f}_s \cdot \underline{e}_u = 0 \qquad \underline{f}_s\, \underline{e}_s = 1$$

$$\rightarrow \quad \alpha = \underline{f}_u \cdot \underline{v} \qquad \beta = \underline{f}_s\, \underline{v}$$

⑨ Wie verhält sich ξ_{n+1} in der Nähe von ξ_F für kleine p um p_0?
(Linearisierung der Map)

$$\xi_{n+1} = p_n g + [\lambda_u e_u f_u + \lambda_s e_s f_s] \cdot (\xi_n - p_n g)$$

Lage der verschobenen FPs

Linearisierung der Map

⑩ + ⑪ + ⑫: Kontrollmethode

- Idee: wähle p_n so, dass ξ_{n+1} auf der stabilen Mannigfalt. des FPs liegt:
  Bedingung:
  $$f_u \cdot \xi_{n+1} = 0$$
- Nach p_n auflösen gibt

$$p_n = \lambda_u (\lambda_u - 1)^{-1} (f_u \cdot \xi_n) / (f_u \cdot g)$$

(13) Kontrollbedingung

- Wir wenden die Kontrolle nur an,

wenn ξ_n nahe genug am FP ist:
und wenn $-p_* < p < p_*$:

$$\xi_n^u := f_u \cdot \underline{\xi_n}$$

$$|\xi_n^u| < \delta \quad \text{(im Paper } \delta = \xi_*)$$

mit

$$\delta := p_* \left|(1 - \lambda_u^{-1})\right| (f_u \cdot g)$$

m

Controlling Chaos

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It is shown that one can convert a chaotic attractor to any one of a large number of possible attracting time-periodic motions by making only *small* time-dependent perturbations of an available system parameter. The method utilizes delay coordinate embedding, and so is applicable to experimental situations in which *a priori* analytical knowledge of the system dynamics is not available. Important issues include the length of the chaotic transient preceding the periodic motion, and the effect of noise. These are illustrated with a numerical example.

PACS numbers: 05.45.+b

The presence of chaos in physical systems has been extensively demonstrated and is very common. In practice, however, it is often desired that chaos be avoided and/or that the system performance be improved or changed in some way. Given a chaotic attractor, one approach might be to make some large and possibly costly alteration in the system which completely changes its dynamics in such a way as to achieve the desired behavior. Here we assume that this avenue is not available. Thus, we address the following question: Given a chaotic at-
1 tractor, how can one obtain improved performance and a desired attracting time-periodic motion by making only *small* time-dependent perturbations in an *accessible* system parameter?

The key observation is that a chaotic attractor typically has embedded within it an infinite number of unstable periodic orbits.[1] Since we wish to make only small perturbations to the system, we do not envision creating new
2 orbits with very different properties from the existing ones. Thus, we seek to exploit the already existing unstable periodic orbits. Our approach is as follows: We first determine some of the unstable low-period periodic orbits that are embedded in the chaotic attractor. We then examine these orbits and choose one which yields improved system performance. Finally, we tailor our small time-dependent parameter perturbations so as to stabilize this already existing orbit. In this Letter we describe how this can be done, and we illustrate the method with a numerical example. The method is very general and should be capable of yielding greatly improved performance in a wide variety of situations.

It is interesting to note that if the situation is such that the suggested method is practical, then the presence of chaos can be a great advantage. The point is that any one of a number of different orbits can be stabilized, and the choice can be made to achieve the best system performance among those orbits. If, on the other hand, the attractor is not chaotic but is, say, periodic, then small parameter perturbations can only change the orbit slightly. Basically we are then stuck with whatever system performance the stable periodic orbit gives, and we have no option for substantial improvement, short of making large alterations in the system.

Furthermore, one may want a system to be used for different purposes or under different conditions at different times. Thus, depending on the use, different requirements are made of the system. If the system is chaotic, this type of multiple-use situation might be accommodated without alteration of the gross system configuration. In particular, depending on the use desired, the system behavior could be changed by switching the temporal programming of the small parameter perturbations to stabilize different orbits. In contrast, in the absence of chaos, completely separate systems might be required for each use. Thus, when designing a system intended for multiple uses, purposely building chaotic dynamics into the system may allow for the desired flexibility. Such multipurpose flexibility is essential to higher life forms, and we, therefore, speculate that chaos may be a necessary ingredient in their regulation by the brain.

To simplify the analysis we consider continuous-time dynamical systems which are *three dimensional* and depend on one system parameter which we denote p [for example, $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x},p)$, where $\mathbf{x}$ is three dimensional]. 3
We assume that the parameter p is available for external adjustment, and we wish to temporally program our adjustments of p so as to achieve improved performance. We emphasize that our restriction to a three-dimensional system is mainly for ease of presentation, and that the case of higher-dimensional (including infinite-dimensional) systems can be treated by similar methods.[2]

We imagine that the dynamical equations describing the system are not known, but that experimental time series of some scalar-dependent variable $z(t)$ can be measured. Using delay coordinates[3,4] with delay T one 4
can form a delay-coordinate vector,

$$\mathbf{X}(t) = [z(t), z(t-T), z(t-2T), \ldots, z(t-MT)] .$$

We are interested in periodic orbits and their stability properties, and we shall use $\mathbf{X}$ to obtain a surface of section for this purpose. In the surface of section, a continuous-time-periodic orbit appears as a discrete-time orbit cycling through a finite set of points. We require the dynamical behavior of the surface of section map in *neighborhoods* of these points in order to study the stability of the periodic orbits. To embed a small neighborhood of a point from $\mathbf{x}$ into $\mathbf{X}$, we typically only require as many dimensions as there are coordinates of the point. Thus, for our purposes, $M=D-1$ is generally sufficient. (This is in contrast with[3] $M+1=2D+1$, typically required for global embedding of the original phase space in the delay-coordinate space.) Hence, for the case considered ($D=3$), our surface of section is two dimensional.

5 We suppose that the parameter p can be varied in a small range about some nominal value p_0. Henceforth, without loss of generality, we set $p_0 \equiv 0$. Let the range in which we are allowed to vary p be $p_* > p > -p_*$.

6 Using an experimental surface of section for the embedding vector $\mathbf{X}$, we imagine that we obtain many experimental points in the surface of section for $p=0$. We denote these points $\xi_1, \xi_2, \xi_3, \ldots, \xi_k$, where ξ_n denotes the coordinates in the surface of section at the nth piercing of the surface of section by the orbit $\mathbf{X}(t)$. For example, a common choice of the surface of section would be $z(t-MT)$ equals a constant, and $\xi_n = [z(t_n), \ldots, z(t_n-(M-1)T)]$, where $t=t_n$ denotes the time at the nth piercing. From such experimentally determined sequences it has been demonstrated that a large number of distinct unstable periodic orbits on a chaotic attractor can be determined.[5,6] We then examine these unstable periodic orbits and select the one which gives the best performance. Again using an experimentally determined sequence, we obtain the stability properties of the chosen periodic orbit (cf. Refs. 5 and 6 for discussion of how this can be done and for descriptions of its implementation in concrete experimental cases). For the purposes of simplicity, let us assume in what follows that this orbit is a fixed point of the surface of section map (i.e., period 7a one; the case of higher period is a straightforward extension). Let λ_s and λ_u be the experimentally determined stable and unstable eigenvalues of the surface of section map at the chosen fixed point of the map ($|\lambda_u| > 1 > |\lambda_s|$). Let $\mathbf{e}_s$ and $\mathbf{e}_u$ be the experimentally determined unit vectors in the stable and unstable directions. 7b Let $\xi = \xi_F \equiv 0$ be the desired fixed point. We then change p slightly from $p=0$ to some other value $p=\bar{p}$. The fixed-point coordinates in the experimental surface of section will shift from 0 to some nearby point $\xi_F(\bar{p})$ and we determine this new position. For small $\bar{p}$ we approximate $\mathbf{g} \equiv \partial \xi_F(p)/\partial p|_{p=0} \cong \bar{p}^{-1} \xi_F(\bar{p})$, which allows an experimental determination of the vector $\mathbf{g}$.

9 Thus, in the surface of section, near $\xi=0$, we can use a linear approximation for the map, $\xi_{n+1} - \xi_F(p) \cong \mathbf{M} \cdot [\xi_n - \xi_F(p)]$, where $\mathbf{M}$ is a 2×2 matrix. Using $\xi_F(p) \cong p\mathbf{g}$ we have

$$\xi_{n+1} \cong p_n \mathbf{g} + [\lambda_u \mathbf{e}_u \mathbf{f}_u + \lambda_s \mathbf{e}_s \mathbf{f}_s] \cdot [\xi_n - p_n \mathbf{g}] . \tag{1}$$

[In the linearization (1), we have considered p_n to be small and of the same order as ξ_n.] We emphasize that $\mathbf{g}$, $\mathbf{e}_u$, $\mathbf{e}_s$, λ_u, and λ_s are all experimentally accessible by the embedding technique just discussed. In (1) $\mathbf{f}_u$ and 8 $\mathbf{f}_s$ are contravariant basis vectors defined by $\mathbf{f}_s \cdot \mathbf{e}_s = \mathbf{f}_u \cdot \mathbf{e}_u = 1$, $\mathbf{f}_s \cdot \mathbf{e}_u = \mathbf{f}_u \cdot \mathbf{e}_s = 0$. Note that we have written the location of the fixed point as $p_n \mathbf{g}$ because we imagine that we adjust p to a new value p_n after each piercing of the surface of section. That is, we observe ξ_n and 10 then adjust p to the value p_n. Thus p_n depends on ξ_n. Further, we only envision making this adjustment when the orbit falls near the desired fixed point for $p=0$.

Assume that ξ_n falls near the desired fixed point at $\xi=0$ so that (1) applies. We then attempt to pick p_n so that ξ_{n+1} falls on the stable manifold of $\xi=0$. That is, we choose p_n so that $\mathbf{f}_u \cdot \xi_{n+1} = 0$. If ξ_{n+1} falls on the 11 stable manifold of $\xi=0$, we can then set the parameter perturbations to zero, and the orbit for subsequent time will approach the fixed point at the geometrical rate λ_s. Thus, for sufficiently small ξ_n, we can dot (1) with $\mathbf{f}_u$ to obtain

$$p_n = \lambda_u (\lambda_u - 1)^{-1} (\xi_n \cdot \mathbf{f}_u)/(\mathbf{g} \cdot \mathbf{f}_u) , \tag{2}$$ 12

which we use when the magnitude of the right-hand side of (2) is less than p_*. When it is greater than p_*, we set $p_n = 0$. We assume in (2) that the generic condition $\mathbf{g} \cdot \mathbf{f}_u \neq 0$ is satisfied. Thus, the parameter perturbations 13 are activated (i.e., $p_n \neq 0$) only if ξ_n falls in a narrow strip $|\xi_n^u| < \xi_*$, where $\xi_n^u = \mathbf{f}_u \cdot \xi_n$, and from (2) $\xi_* = p_* |(1-\lambda_u^{-1}) \mathbf{g} \cdot \mathbf{f}_u|$. Thus, for small p_*, a typical initial condition will execute a chaotic orbit, unchanged from the uncontrolled case, until ξ_n falls in the strip. Even then, because of nonlinearity not included in (1), the control may not be able to bring the orbit to the fixed point. In this case the orbit will leave the strip and continue to wander chaotically as if there was no control. Since the orbit on the uncontrolled chaotic attractor is ergodic, at some time it will eventually satisfy $|\xi_n^u| < \xi_*$ and also be sufficiently close to the desired fixed point that attraction to $\xi=0$ is achieved. [In rare cases applying Eq. (2) when the trajectory enters the strip, but is still far from 0, may result in stabilizing the wrong periodic orbit which visits the strip.]

Thus, we create a stable orbit, but, for a typical initial condition, it is preceded in time by a chaotic transient in which the orbit is similar to orbits on the uncontrolled chaotic attractor. The length τ of such a chaotic transient depends sensitively on the initial condition, and, for randomly chosen initial conditions, has an exponential probability distribution[7] $P(\tau) \sim \exp[-(\tau/\langle\tau\rangle)]$ for large τ. The average length of the chaotic transient $\langle\tau\rangle$ increases with decreasing p_* and follows a power-law relation[7] for small p_*, $\langle\tau\rangle \sim p_*^{-\gamma}$.

We will now derive a formula for the exponent γ. Dotting the linearized map for ξ_{n+1}, Eq. (1), with $\mathbf{f}_u$, we obtain $\xi_{n+1}^u \cong 0$. In obtaining this result from (1) we have substituted p_n appropriate for $|\xi_n^u| < \xi_*$. We note that the result $\xi_{n+1}^u \cong 0$ is a linearization, and typically