Synchronization in Nonlinear Systems and Networks

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Wednesday 13:00-14:00

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Attractor $A$ is a limiting set in phase space, towards which a dynamical system evolves over time.

This limiting set $A$ can be:

1) point (equilibrium)
2) curve (periodic orbit)
3) torus (quasiperiodic orbit)
4) fractal (*strange attractor* = CHAOS)
Attractor

is a closed set with the following properties:

1. \(A\) is an *invariant set*: any trajectory \(x(t)\) that starts in \(A\) stays in \(A\) for all time.

2. \(A\) *attracts an open set of initial conditions*: there is an open set \(U\) containing \(A\) such that if \(x(0) \in U\), then the distance from \(x(t)\) to \(A\) tends to zero as \(t \to \infty\). This means that \(A\) attracts all trajectories that start sufficiently close to it. The largest such \(U\) is called the *basin of attraction* of \(A\).

3. \(A\) is *minimal*: there is no proper subset of \(A\) that satisfies conditions 1 and 2.

*S. Strogatz*
The Lorenz attractor is generated by the system of 3 differential equations

\[
\begin{align*}
\frac{dx}{dt} &= -10x + 10y \\
\frac{dy}{dt} &= 28x - y - xz \\
\frac{dz}{dt} &= -\frac{8}{3}z + xy.
\end{align*}
\]
What can we learn from the two exemplary 3-dimensional flow?

If a flow is **locally unstable** in each point but **globally bounded**, any open ball of initial points will be stretched out and then folded back.

If the equilibria are hyperbolic, the trajectory will be attracted along some eigen-directions and ejected along others.
Reducing to discrete dynamics. Lorenz map

Continues dynamics. Variable $z(t)$

Lorenz one-dimensional map

$z_{n+1} = f(z_n)$
**Poincare section and Poincare return map**

**Figure 3.6:** Return maps for the $R_n \rightarrow R_{n+1}$ radial distance Poincaré sections of figure 3.5. (R. Paškauskas)
Tent map and logistic map

\[ f(x) = \begin{cases} 
r x, & 0 \leq x \leq \frac{1}{2} \\
 r - r x, & \frac{1}{2} \leq x \leq 1 
\end{cases} \]

\[ x_{n+1} = r x_n (1 - x_n) \]
How common is chaos in dynamical systems?

To answer the question, we need discrete dynamical systems given by

one-dimensional maps
Simplest examples of chaotic maps

\[ x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots \]

\[ f(x) = 2x \mod 1 \] (Bernoulli map)

\[ f(x) = \frac{a}{2} - a|x - \frac{1}{2}| \] (tent map)

\[ f(x) = ax(1-x) \] (logistic map)
Bifurcation diagram of tent map: Chaotic dynamics

- one-band chaotic attractor
- 8-band
- 4-band
- 2-band
- homoclinic bifurcation

 参数：μ
Magnification near the tip shows separated chaotic regions.
Bifurcation diagram of logistic map: Regular versus chaotic dynamics

\[ f(x) = ax(1 - x), \quad x \in [0,1], \quad \text{parameter } a \in [0,4] \]
Periodic windows of the logistic map

(up to period 8)
Period-3 window

Suddenly, at \( a = 3.8284 \) a period-3 cycle \( P_3 \) arises out of the blue sky and lasts till \( a = 3.8415 \). Window \( W_3 = [3.8284..; 3.8415..] \)

Graph of \( f^3(x) = f(f(f(x))) \)

Stable period-3 cycle \( P_3 \) is born in a tangent (or saddle-node) bifurcation
Moment of tangent bifurcation of $P_3$: parameter $a = 1 + \sqrt{8} = 3.8284 \ldots$
Intermittency route to chaos

Just before the tangent bifurcation of period-3 cycle $P_3$
Cascade of homoclinic bifurcations

\[ a_0 = 3.678\,573\,510\,428\,32 \ldots, \]
\[ a_1 = 3.592\,572\,184\,106\,97 \ldots, \]
\[ a_2 = 3.574\,804\,938\,759\,20 \ldots, \]
\[ a_3 = 3.570\,985\,940\,341\,61 \ldots . \]

\[ \delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669 \ldots \]

\[ \frac{d_n}{d_{n+1}} \to \alpha = -2.5029 \ldots \]

**Sharkovsky ordering (1964)**

For any continuous 1-Dim map, periods of cycles (periodic orbits) are ordered as:

\[ 1 < 2 < 2^2 < 2^3 < \ldots < 2^n \ldots \ldots < 7 \cdot 2^n < 5 \cdot 2^n < 3 \cdot 2^n \ldots \]

\[ \ldots < 7 \cdot 2 < 5 \cdot 2 < 3 \cdot 2 \ldots < 9 < 7 < 5 < 3. \]
“Period three implies chaos” (Li, Yorke 1975)

**Theorem 1.** Let \( J \) be an interval and let \( F: J \to J \) be continuous. Assume there is a point \( a \in J \) for which the points \( b = F(a) \), \( c = F^2(a) \) and \( d = F^3(a) \), satisfy

\[ d \leq a < b < c \quad \text{(or } d \geq a > b > c \text{)} . \]

Then

**T1:** for every \( k = 1, 2, \cdots \) there is a periodic point in \( J \) having period \( k \).

Furthermore,

**T2:** there is an uncountable set \( S \subset J \) (containing no periodic points), which satisfies the following conditions:

(A) For every \( p, q \in S \) with \( p \neq q \),

\[ \lim_{n \to \infty} \sup \left| F^n(p) - F^n(q) \right| > 0 \quad \text{(2.1)} \]

and

\[ \lim_{n \to \infty} \inf \left| F^n(p) - F^n(q) \right| = 0. \quad \text{(2.2)} \]
Lyapunov exponent for logistic map.

\[ f_a : x \rightarrow ax(1-x), \quad a \in [0,4] \]

Bifurcation diagram

Lyapunov exponent \( \lambda \) is positive on a nowhere dense Cantor-like set \( K \) of parameter \( a \), and \( \text{mes } K > 0 \)
Strange attractor in Henon map (1976)

\[ x_{n+1} = 1 - ax_n^2 + by_n \]
\[ y_{n+1} = x_n \]
\[ a = 1.4, \quad b = 0.3 \]

Lozi map (1976)

\[ x_{n+1} = 1 - a|x_n| + by_n \]
\[ y_{n+1} = x_n . \]