English summary

1.2 Bifurcations (continued)

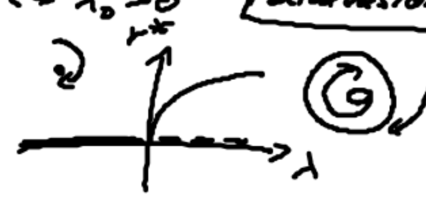
(B) Hopf bifurcation: $\lambda_{1,2} = \lambda_0 \pm i\omega$, $\lambda_0 < 0 \leftrightarrow \lambda_0 > 0$ $\boxed{\text{dimension } n=2}$

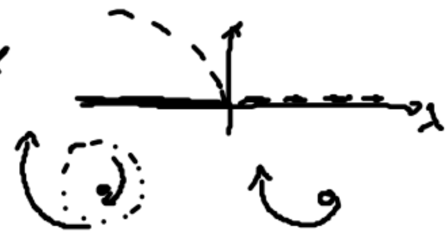
$z = x + iy = re^{i\varphi} \in \mathbb{C}$

supercritical

$$\dot{z} = \left(\lambda + i\omega \mp (1+i\gamma)|z|^2\right) z$$

subcritical





(C) local bifurcations of limit cycles

(C1) saddle-node bifurcation of limit cycles
fold bifurcation

(C2) period-doubling bifurcation
flip bifurcation

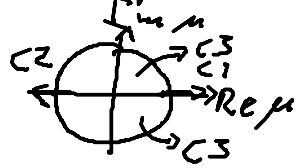
(C3) Neimarck-Sacker bifurcation
(secondary Hopf bif.)

1^{st} Hopf (of FP): $\phi_1 = \omega_1 t$ $\quad \frac{\omega_1}{\omega_2} \in \mathbb{Q} \Rightarrow$ closed orbit

2^{nd} Hopf (of LC): $\phi_2 = \omega_2 t$ $\quad \frac{\omega_1}{\omega_2} \notin \mathbb{Q} \Rightarrow$ quasiperiodicity

Floquet multiplier: $\mu = e^{\Lambda T} \in \mathbb{C}$

stability: $|\mu| \gtrless 1$?



(D) global bifurcations of limit cycles:

(D1) saddle-node bifurcation on a limit cycle

SNIPER/SNIC

→ Chap. 2.2

2 Phänomenologische Modelle

Mathematische Beschreibung neuronaler Dynamik:

- Physiological models: (Kap. 3)
  - e.g. Hodgkin-Huxley equations
  - Many physiological details and processes
  - Detailed description of single cell
  - Many equations, many parameters
  - Applicable to ensembles of many oscillators?
  - Feasible for bifurcation analysis?

- Normal-form equations:
  - e.g. FitzHugh-Nagumo equations, SNIC/SNIPER equations
  - Main dynamical behavior (type of excitability)
  - Feasible for bifurcation analysis
  - Few equations, few parameters
  - Applicable to ensembles of many oscillators
  - Detailed description of single cell?
  - Physiological relevant processes?

2.1 FitzHugh-Nagumo-Modell
2.2 SNIPER Modell
2.3 Hindmarsh-Rose-Modell

2.1 FitzHugh-Nagumo-Modell

Dynamische Gleichungen:

$$\varepsilon \dot{x} = x - \frac{x^3}{3} - y \qquad \text{dynamische Variablen:}$$

$$\dot{y} = x + a$$

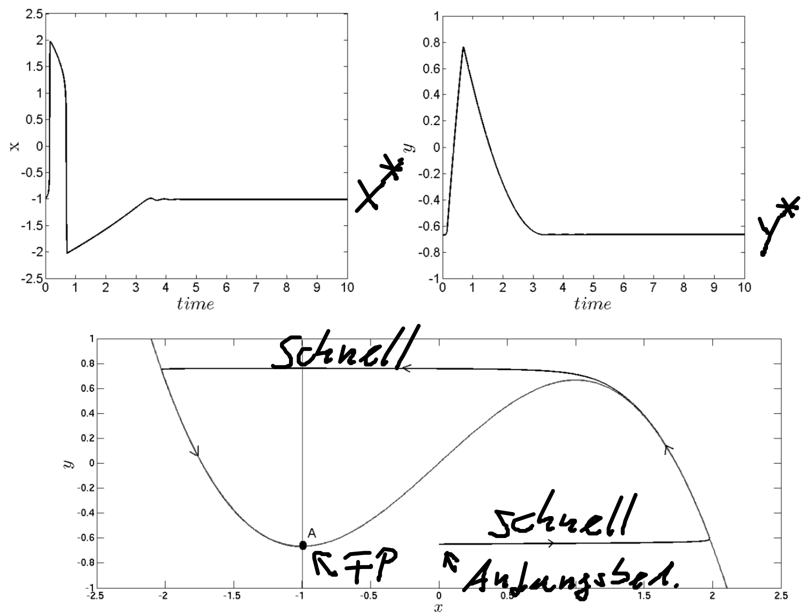
dynamische Variablen:
x : Aktivator
y: Inhibitor

Parameter: a : Bifurkation
ε : Zeitskalentrennung ($\varepsilon \ll 1$)
(x: schnell, $\dot{x}$ groß; y: langsam, $\dot{y}$ klein)

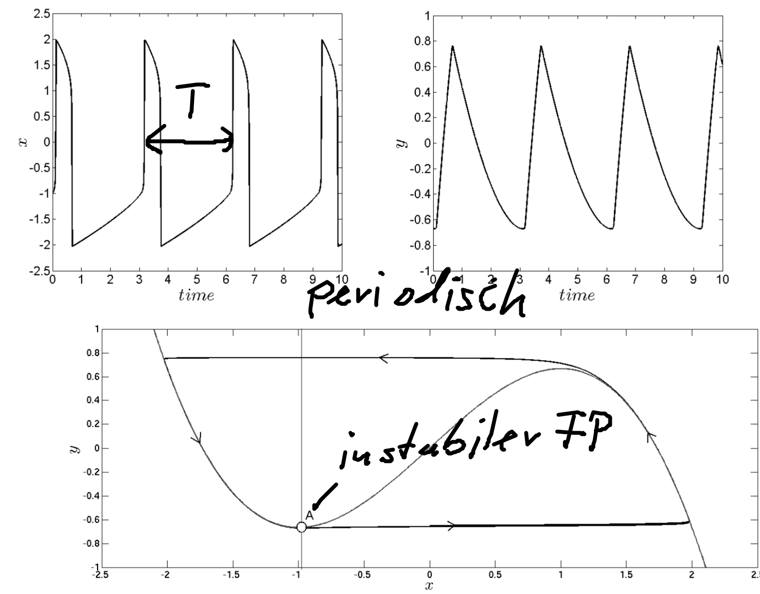
a bestimmt das dynamische Verhalten:

$|a| < 1$: oszillierend (Grenzzyklus)

$|a| > 1$: anregbar (Fixpunkt)



$\varepsilon = 0{,}005,\ a = 1.01 > 1$



$\varepsilon = 0.005,\ a = 0.97 < 1$

Bestimmung des Fixpunkts (x^*, y^*):

Bed.: $\dot{x} = 0,\ \dot{y} = 0 \Rightarrow$ Schnittpunkt der Nullklinen:

$$\left.\begin{array}{l} 0 = x - \frac{x^3}{3} - y \\ 0 = x + a \end{array}\right\} \Rightarrow x^* = -a \Rightarrow y^* = x^* - \frac{(x^*)^3}{3} = -a + \frac{a^3}{3}$$

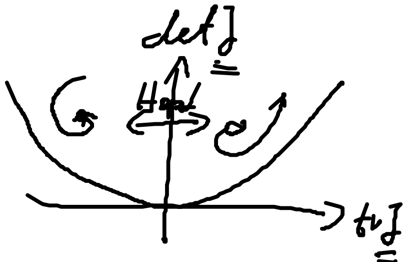
Bestimmung der Stabilität / Stabilitätsanalyse:

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \end{pmatrix} = \left.\begin{pmatrix} \frac{1-x^2}{\epsilon} & -\frac{1}{\epsilon} \\ 1 & 0 \end{pmatrix}\right|_{x^*,y^*} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1-a^2}{\epsilon} & -\frac{1}{\epsilon} \\ 1 & 0 \end{pmatrix}}_{=\underline{\underline{J}}} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

Eigenwerte der Jacobi-Matrix:

$$0 = \det \begin{pmatrix} \frac{1-a^2}{\epsilon} - \lambda & -\frac{1}{\epsilon} \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda \, \mathrm{tr}\,\underline{\underline{J}} + \det \underline{\underline{J}}$$

$$= \lambda^2 - \frac{1-a^2}{\epsilon}\lambda + \frac{1}{\epsilon} \Rightarrow \lambda_{1,2} = \frac{1-a^2 \pm \sqrt{(1-a^2)^2 - 4\epsilon}}{2\epsilon}$$



$$\mathrm{tr}\,\underline{\underline{J}} = \frac{1-a^2}{\epsilon} \begin{cases} > 0 & \text{für } |a| < 1 \\ < 0 & \text{für } |a| > 1 \end{cases}$$

$$\det \underline{\underline{J}} = \frac{1}{\epsilon} > 0$$

$\Rightarrow$ stabiler FP ($\mathrm{Re}\,\lambda < 0$) für $|a| > 1$

instabiler FP ($\mathrm{Re}\,\lambda > 0$) für $|a| < 1$

Bifurkationspunkt: $|a| = 1$

$\Rightarrow \mathrm{Im}\,\lambda\big|_{|a|=1} = \frac{1}{\sqrt{\epsilon}} \Rightarrow$ endliche Frequenz

$\Rightarrow$ Anregbarkeit Typ $\underline{\mathrm{II}}$

Fazit: Der Fixpunkt verliert seine Stabilität in einer Hopf-Bifurkation

Erweitertes / ursprüngliches FitzHugh-Nagumo-System

$$\epsilon \dot{x} = x - \frac{x^3}{3} - y$$

$$\dot{y} = x + a - \gamma y$$

IMPULSES AND PHYSIOLOGICAL STATES IN THEORETICAL MODELS OF NERVE MEMBRANE

Biophys. J 1, 445 (1961)

RICHARD FITZHUGH
From the National Institutes of Health, Bethesda

The following linear differential equation describes an oscillating quantity x with damping constant k (the dots represent differentiation with respect to time t):

$$\ddot{x} + k\dot{x} + x = 0$$

Van der Pol (1926) replaced the damping constant by a damping coefficient which depends quadratically on x:

$$\ddot{x} + c(x^2 - 1)\dot{x} + x = 0$$

where c is a positive constant. It is convenient to use Liénard's transformation (Liénard, 1928; Minorsky, 1947):

$$y = \dot{x}/c + x^3/3 - x$$

and obtain the following pair of differential equations:

$$\dot{x} = c(y + x - x^3/3)$$

$$\dot{y} = -x/c$$

The BVP model is obtained by adding terms to these equations as follows:—

$$\dot{x} = c(y + x - x^3/3 + z) \tag{1}$$

$$\dot{y} = -(x - a + by)/c \tag{2}$$

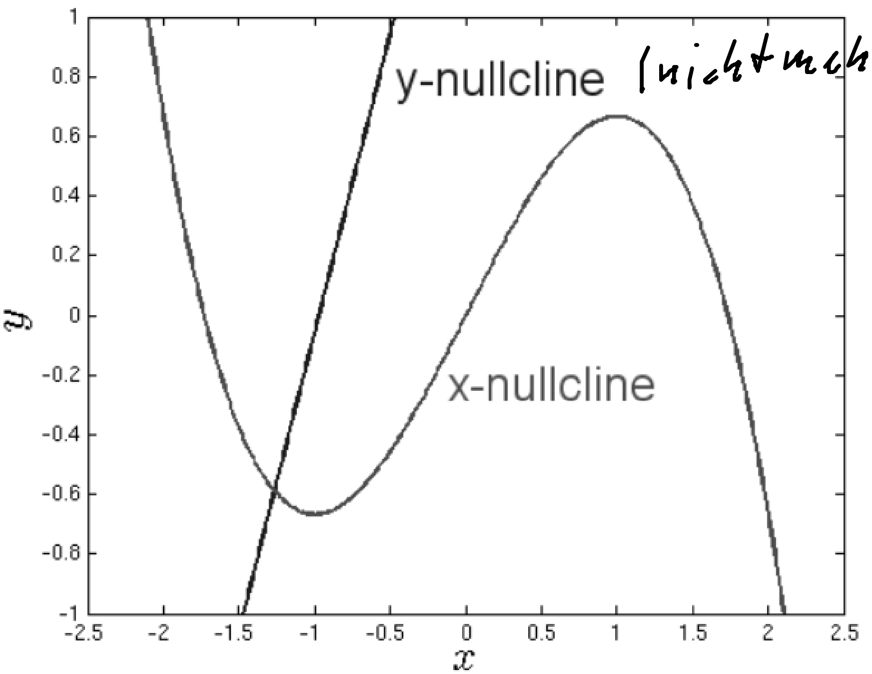
1962 *PROCEEDINGS OF THE IRE* 2061

An Active Pulse Transmission Line Simulating Nerve Axon*

J. NAGUMO†, MEMBER, IRE, S. ARIMOTO†, AND S. YOSHIZAWA†

Recently, FitzHugh ingeniously simplified the H-H equations in case of a "space clamp," making use of an analog computer, and proposed the following BVP model (Bonhoeffer-van der Pol model).[9]

$$\begin{cases} J = \dfrac{1}{c}\dfrac{du}{dt} - w - \left(u - \dfrac{u^3}{3}\right), \\ c\dfrac{dw}{dt} + bw = a - u, \end{cases} \tag{2}$$



(nicht mehr senkrecht für $\gamma \neq 0$)

$\dot{y} = 0 \Rightarrow y = \frac{x}{\gamma} + \frac{a}{\gamma}$

hier: $a = 0.97$

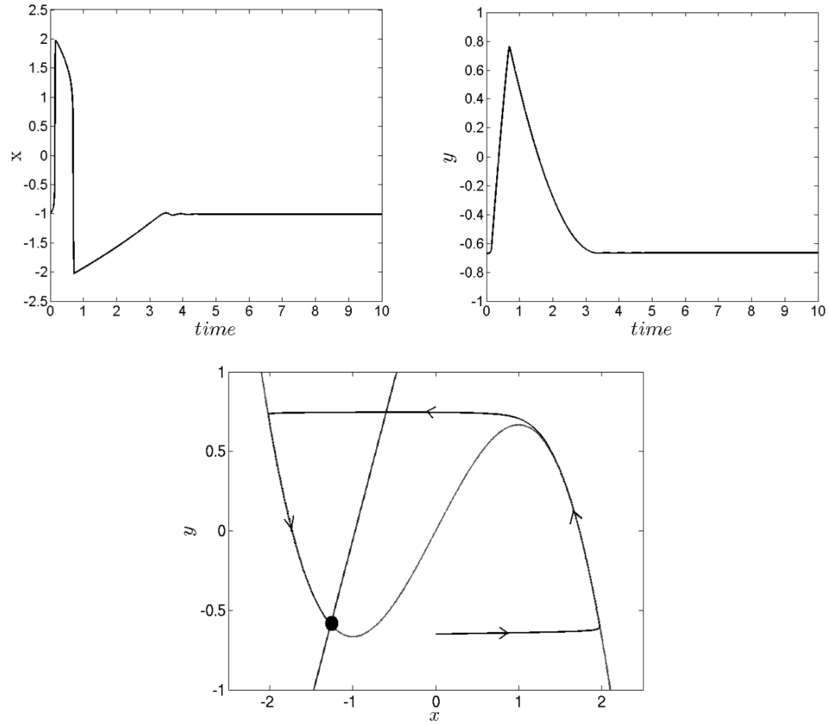
$\gamma = 0.5$

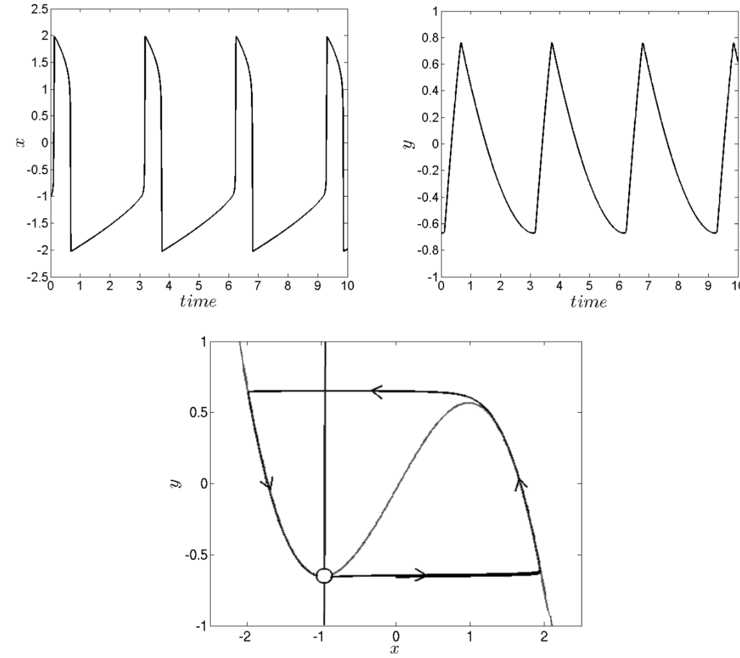
Bestimmung von (x^*, y^*): $0 = -\frac{a}{\gamma} + \left(1 - \frac{1}{\gamma}\right)x^* - \frac{(x^*)^3}{3}$

Nullstellen einer kubischen Gleichung

$$\begin{pmatrix}\delta\dot{x}\\ \delta\dot{y}\end{pmatrix} = \begin{pmatrix}\frac{1-(x^*)^2}{\epsilon} & -\frac{1}{\epsilon}\\ 1 & -\gamma\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \Rightarrow \lambda_{1,2} = \frac{\mathrm{tr}\,\underline{\underline{J}} \pm \sqrt{(\mathrm{tr}\,\underline{\underline{J}})^2 - 4\det\underline{\underline{J}}}}{2}$$

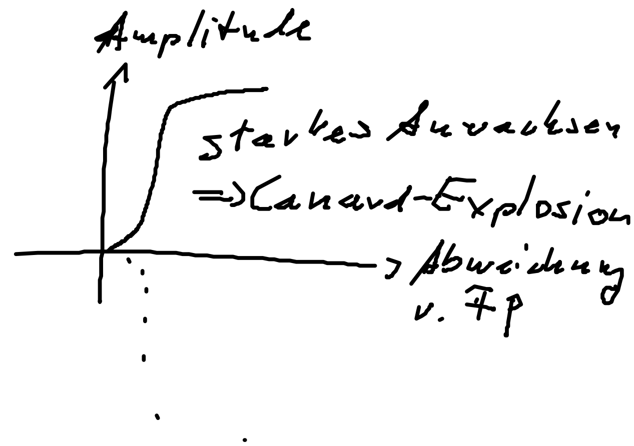
Rolle von Parameter γ:





oszillierend: $\varepsilon = 0.005$, $a = 0.97$
$\gamma = 0.005$

anregbar: $\varepsilon = 0.005$, $a = 0.97$, $\gamma = 0.5$



$a = 0.67$
$\gamma = 0.5$