

English summary

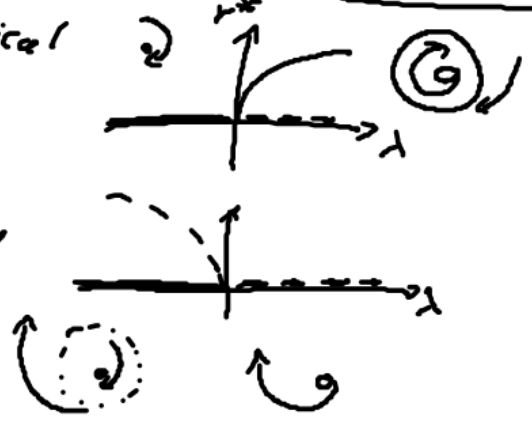
1.2 Bifurcations (continued)

(B) Hopf bifurcation: $\lambda_{1,2} = \lambda_0 \pm i\omega$, $\lambda_0 < 0 \leftrightarrow \lambda_0 > 0$
 $z = x + iy = r e^{i\varphi} \in \mathbb{C}$

dimension $n=2$

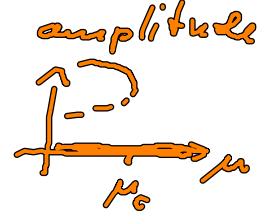
$$\dot{z} = (\lambda + i\omega \mp (1+i\gamma)|z|^2)z$$

← supercritical
 ← subcritical

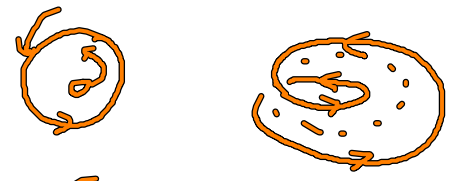


(C) local bifurcations of limit cycles

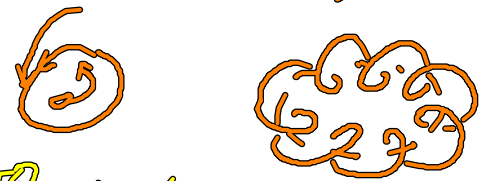
(C1) saddle-node bifurcation of limit cycles
 fold bifurcation



(C2) period-doubling bifurcation
 flip bifurcation

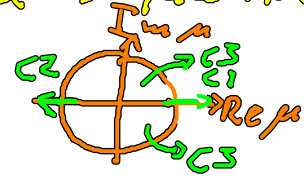


(C3) Neimark-Sacker bifurcation
 (secondary Hopf bif.)



1st Hopf (of FP): $\phi_1 = \omega_1 t$
 2nd Hopf (of LC): $\phi_2 = \omega_2 t$
 $\left. \begin{array}{l} \omega_1/\omega_2 \in \mathbb{Q} \Rightarrow \text{closed orbit} \\ \omega_1/\omega_2 \notin \mathbb{Q} \Rightarrow \text{quasiperiodicity} \end{array} \right\}$

Floquet multiplier: $\mu = e^{\lambda T} \in \mathbb{C}$
 stability: $|\mu| \gtrless 1$?



(D) global bifurcations of limit cycles:

(D1) saddle-node bifurcation on a limit cycle

SNLPER/SNIC

→ Chap. 2.2



2 Phänomenologische Modelle

Mathematische Beschreibung neuronaler Dynamik:

- ▶ Physiological models: (Kap. 3)
 - ▶ e.g. Hodgkin-Huxley equations
 - ▶ Many physiological details and processes
 - ▶ Detailed description of single cell
 - ▶ Many equations, many parameters
 - ▶ Applicable to ensembles of many oscillators?
 - ▶ Feasible for bifurcation analysis?

- ▶ Normal-form equations:
 - ▶ e.g. FitzHugh-Nagumo equations, SNIC/SNIPER equations
 - ▶ Main dynamical behavior (type of excitability)
 - ▶ Feasible for bifurcation analysis
 - ▶ Few equations, few parameters
 - ▶ Applicable to ensembles of many oscillators
 - ▶ Detailed description of single cell?
 - ▶ Physiological relevant processes?

2.1 FitzHugh-Nagumo-Modell

2.2 SNIPER Modell

2.3 Hindmarsh-Rose-Modell

2.1 FitzHugh-Nagumo-Modell

Dynamische Gleichungen:

$$\epsilon \dot{x} = x - \frac{x^3}{3} - y$$

dynamische Variablen:

x : Aktivator

$$\dot{y} = x + a$$

y : Inhibitor

Parameter, a : Bifurkation

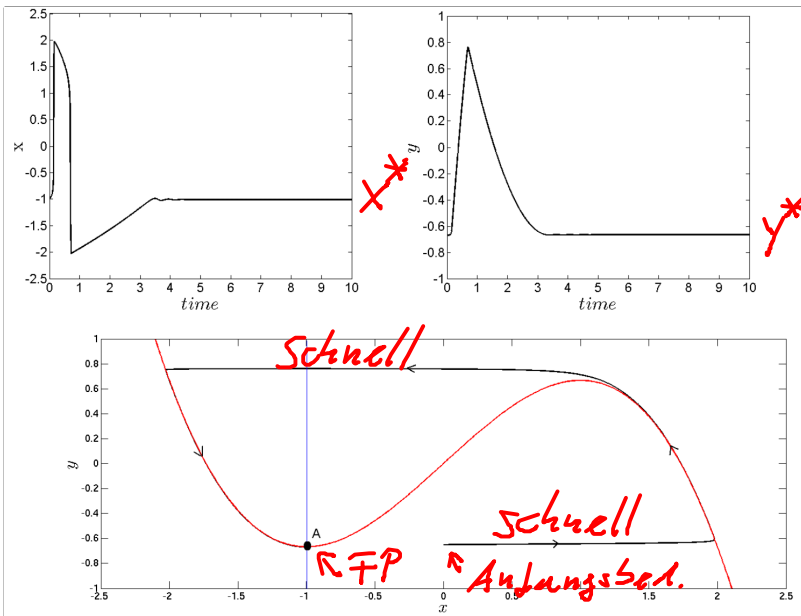
ϵ : Zeitskalentrennung ($\epsilon \ll 1$)

(x : schnell, x groß; y : langsam, y klein)

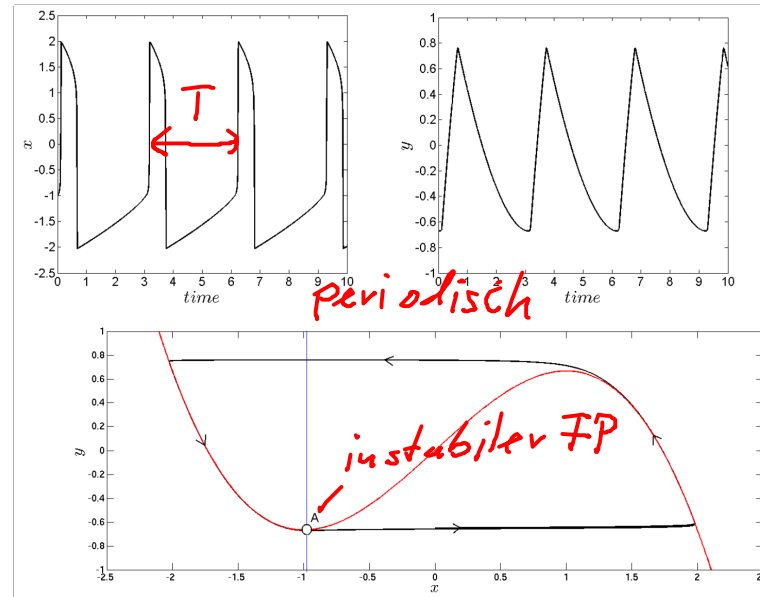
a bestimmt das dynamische Verhalten:

$|a| < 1$: oszillierend (Grenzzyklus)

$|a| > 1$: anregbar (Fixpunkt)



$$\epsilon = 0.005, a = 1.01 > 1$$



$$\epsilon = 0.005, a = 0.37 < 1$$

Bestimmung des Fixpunkts (x^*, y^*):

Bed.: $\dot{x} = 0, \dot{y} = 0 \Rightarrow$ Schnittpunkt der Nullklinen:

$$\left. \begin{aligned} 0 &= x - \frac{x^3}{3} - y \\ 0 &= x + a \end{aligned} \right\} \Rightarrow x^* = -a \Rightarrow y^* = x^* - \frac{(x^*)^3}{3} = -a + \frac{a^3}{3}$$

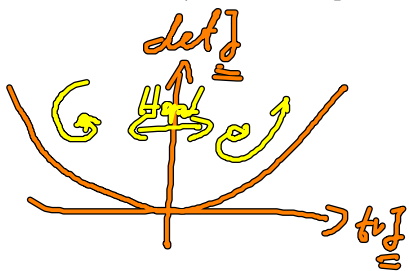
Bestimmung der Stabilität / Stabilitätsanalyse:

$$\begin{pmatrix} \delta \dot{x} \\ \delta \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{1-x^2}{\epsilon} & -\frac{1}{\epsilon} \\ 1 & 0 \end{pmatrix} \Big|_{x^*, y^*} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1-a^2}{\epsilon} & -\frac{1}{\epsilon} \\ 1 & 0 \end{pmatrix}}_{= \underline{J}} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

Eigenwerte der Jacobi-Matrix:

$$0 = \det \begin{pmatrix} \frac{1-a^2}{\epsilon} - \lambda & -\frac{1}{\epsilon} \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda \underbrace{\text{tr} \underline{J}} + \det \underline{J}$$

$$= \lambda^2 - \frac{1-a^2}{\epsilon} \lambda + \frac{1}{\epsilon} \Rightarrow \lambda_{1,2} = \frac{1-a^2 \pm \sqrt{(1-a^2)^2 - 4\epsilon}}{2\epsilon}$$



$$\text{tr} \underline{J} = \frac{1-a^2}{\epsilon} \begin{cases} > 0 & \text{für } |a| < 1 \\ < 0 & \text{für } |a| > 1 \end{cases}$$

\Rightarrow stabiler FP ($\text{Re} \lambda < 0$) für $|a| > 1$

instabiler FP ($\text{Re} \lambda > 0$) für $|a| < 1$

Bifurkationspunkt: $|a| = 1$

$\Rightarrow \text{Im} \lambda \Big|_{|a|=1} = \frac{1}{\sqrt{\epsilon}} \Rightarrow$ unendliche Frequenz

\Rightarrow Anregbarkeit Typ II

Fazit: Der Fixpunkt verliert seine Stabilität in einer Hopf-Bifurkation

Erweitertes / ursprüngliches FitzHugh-Nagumo-System

$$\epsilon \dot{x} = x - \frac{x^3}{3} - y$$

$$\dot{y} = x + a - \gamma y$$

IMPULSES AND PHYSIOLOGICAL STATES IN THEORETICAL MODELS OF NERVE MEMBRANE

Biophys. J, 445

(1961)

RICHARD FITZHUGH

From the National Institutes of Health, Bethesda

The following linear differential equation describes an oscillating quantity x with damping constant k (the dots represent differentiation with respect to time t):

$$\ddot{x} + k\dot{x} + x = 0$$

Van der Pol (1926) replaced the damping constant by a damping coefficient which depends quadratically on x :

$$\ddot{x} + c(x^2 - 1)\dot{x} + x = 0$$

where c is a positive constant. It is convenient to use Liénard's transformation (Liénard, 1928; Minorsky, 1947):

$$y = \dot{x}/c + x^3/3 - x$$

and obtain the following pair of differential equations:

$$\dot{x} = c(y + x - x^3/3)$$

$$\dot{y} = -x/c$$

The BVP model is obtained by adding terms to these equations as follows:—

$$\dot{x} = c(y + x - x^3/3 + z) \quad (1)$$

$$\dot{y} = -(x - a + by)/c \quad (2)$$

1962

PROCEEDINGS OF THE IRE

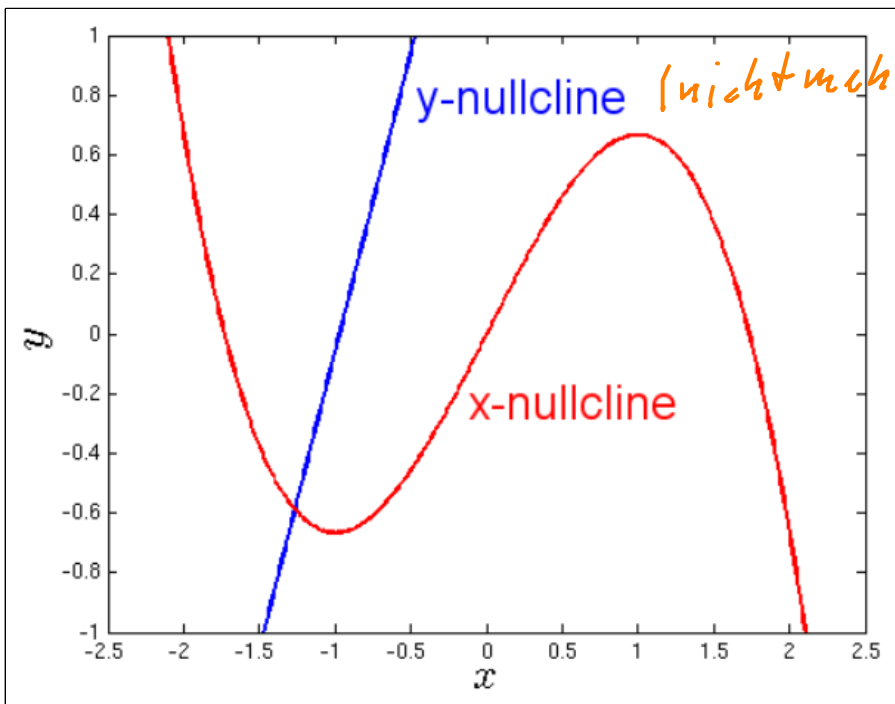
2061

An Active Pulse Transmission Line Simulating Nerve Axon*

J. NAGUMO†, MEMBER, IRE, S. ARIMOTO†, AND S. YOSHIZAWA†

Recently, FitzHugh ingeniously simplified the H-H equations in case of a "space clamp," making use of an analog computer, and proposed the following BVP model (Bonhoeffer-van der Pol model).⁹

$$\begin{cases} J = \frac{1}{c} \frac{du}{dt} - w - \left(u - \frac{u^3}{3} \right), \\ c \frac{dw}{dt} + bw = a - u, \end{cases} \quad (2)$$



senkrecht für $\gamma \neq 0$

$$\dot{y} = 0 \Rightarrow y = \frac{x}{\gamma} + \frac{a}{\gamma^2}$$

$$\text{hier: } a = 0.97$$

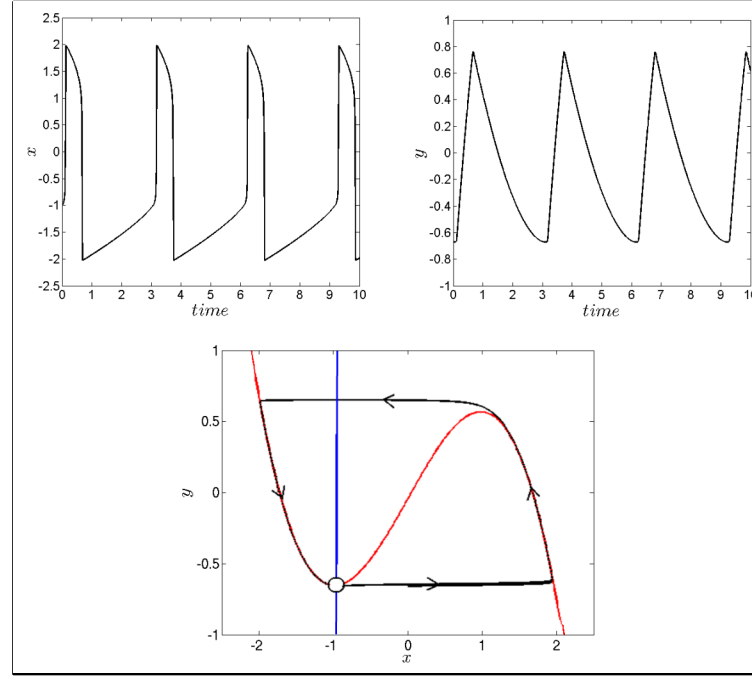
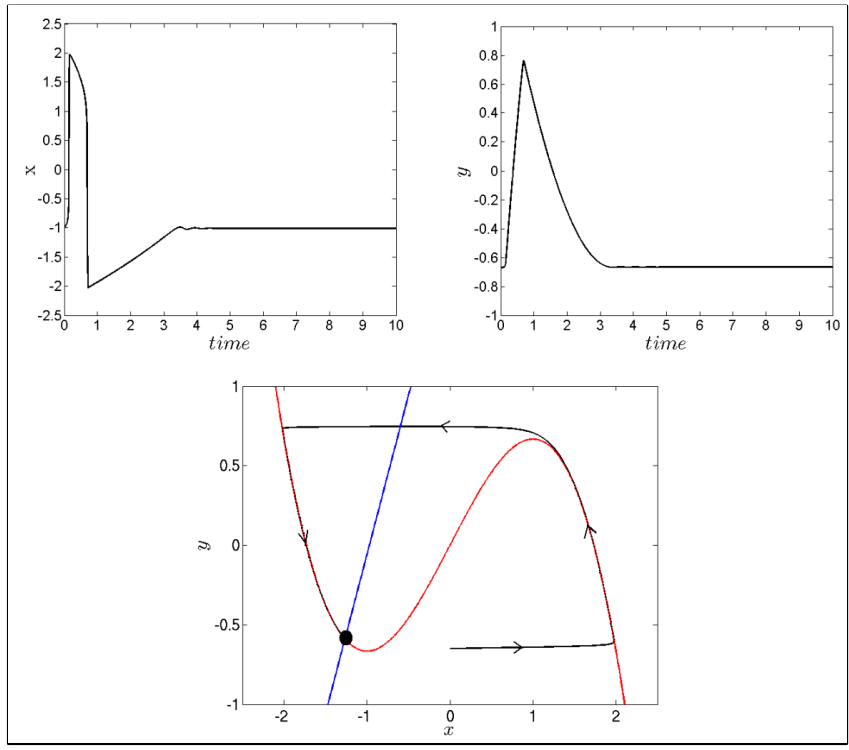
$$\gamma = 0.5$$

Bestimmung von (x^*, y^*) : $0 = -\frac{a}{\gamma} + \left(1 - \frac{1}{\gamma}\right)x^* - \frac{(x^*)^3}{3}$

Nullstellen einer kubischen Gleichung

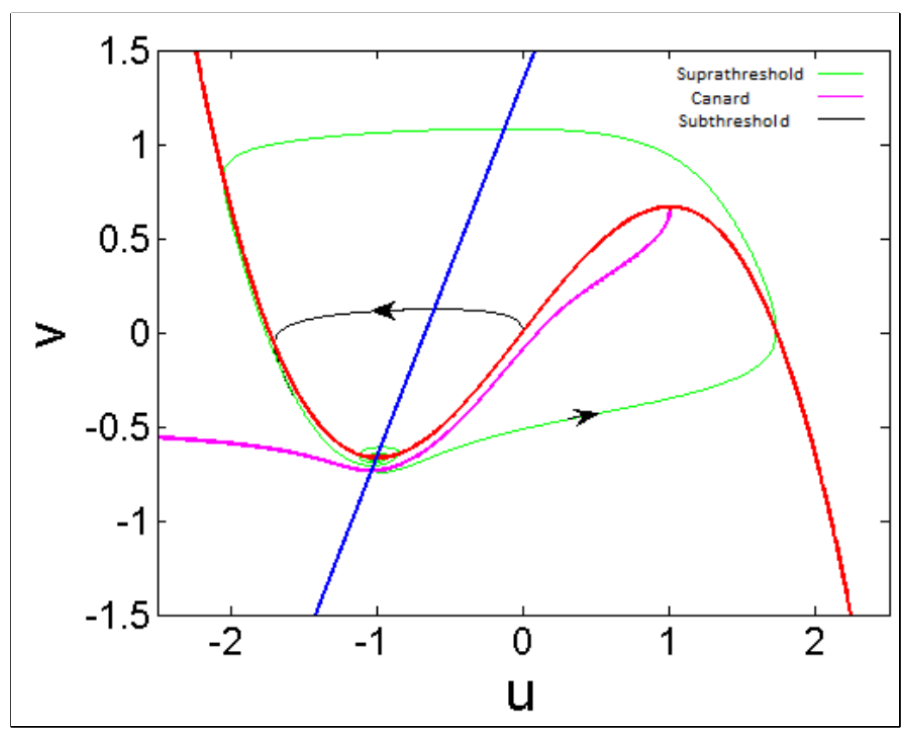
$$\begin{pmatrix} \delta \dot{x} \\ \delta \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{1 - (x^*)^2}{c} & -\frac{1}{c} \\ 1 & -\gamma \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \Rightarrow \lambda_{1,2} = \frac{\tau \pm \sqrt{(\tau \pm 1)^2 - 4 \det J}}{2}$$

Rolle von Parameter γ :



ausregbar: $\epsilon = 0.005, \alpha = 0.97, \gamma = 0.5$

oszillierend: $\epsilon = 0.005, \alpha = 0.97, \gamma = 0.005$



Amplitude
 ↑
 starkes Anwachsen
 ⇒ Canard-Explosion
 → Abweichung v. FP
 α = 0.67
 γ = 0.5