Synchronization and Chimera States in Complex Networks

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Lecture 4

You can find me in room ER 222

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3-Dim Kuramoto model

THE MODEL: \( N \times N \times N \) oscillators inside 3D cube, with periodic boundary conditions

\[
\dot{\varphi}_{i,j,k} = \frac{1}{R^3} \sum_{(i'-i)^2 + (j'-j)^2 + (k'-k)^2 < R^2} \sin(\varphi_{i',j',k'} - \varphi_{i,j,k} - \alpha)
\]

Each oscillator \( \varphi_{i,j,k} \) is coupled with equal strength to all its nearest neighbors \( \varphi_{i',j',k'} \) within range \( R \) regarding the 3D-torus structure of the network.
Regular scroll waves

$\alpha = 0.9 \quad r = 0.16 \quad N = 100 \quad t = 2.0$

$X = 0.5 \quad \text{(red)} \quad Y = 0.25 \quad \text{(green)} \quad Z = 0.25 \quad \text{(blue)}$
Chaotic scroll waves
Synchronous oscillations (no chimeras)
Multistability in Kuramoto model

40 trials with random initial conditions, simulation time \( t = 5000 \), size 50x50x50

<table>
<thead>
<tr>
<th>State</th>
<th>( A_1(r = 0.14; , \alpha = 0.8) )</th>
<th>( A_2(r = 0.16; , \alpha = 0.9) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two-rolls stationary</td>
<td>3</td>
<td>-</td>
</tr>
<tr>
<td>Two rolls travelling</td>
<td>-</td>
<td>6</td>
</tr>
<tr>
<td>Four rolls parallel</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>Four rolls transversal</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>Solitary vortex</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>Two rolls chaotic</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Coherent + close</td>
<td>29 (14 + 15)</td>
<td>27 (18 + 9)</td>
</tr>
<tr>
<td>Rotating wave +close</td>
<td>6 (2 + 4)</td>
<td>4 (3 + 1)</td>
</tr>
</tbody>
</table>

Characteristic regimes of heart disease:

Stationary scroll waves 17.5%

Scroll chaos 2.5%

Coherent wave 82.5%

Normal sinus rhythm
Ventricular tachycardia
Ventricular fibrillation
Deep Brain Stimulation (Benabid, 1986)

permanent high-frequency stimulation >90 Hz

Tremor amplitude as a function of DBS frequency

From: Merrill J. Birdno and Warren M. Grill
Invited review
Basal ganglia local field potential activity: Character and functional significance in the human

Peter Brown*, David Williams

*Sherrill Department of Motor Neuroscience and Movement Disorders, Institute of Neurology, Queen Square, London WCIN
Spontaneous synchronization of coupled limit-cycle oscillators.

We are interested in **spontaneous synchronization** of populations of biological oscillators. For example:

- the pacemaker region of the heart consists of thousands of cells that produce a regular, collective rhythm of electrical signals.

- fireflies, males congregate in a tree and flash in synchrony to the sending signals to their lady friends.
Winfree model (1967)

A population of interacting limit-cycle oscillators:
- weak coupling
- nearly identical oscillators

The oscillators relax to their limit cycles and so, can be characterized solely by their phases

Each oscillator is coupled to the collective rhythm generated by the whole population

\[
\frac{d\psi_i}{dt} = \omega_i + R(\psi_i) \sum_{j=1}^{N} \Gamma(\psi_j), \quad i = 1, \ldots, N
\]

\(\psi_i\) - phases of individual oscillators
\(\omega_i\) - frequencies of individual oscillators
\(\Gamma(\cdot)\) - coupling function

"Biological rhythms and the behavior of populations of coupled oscillators"
J.Theor. Biol. 1967

Book 2000
Arthur Winfree (1942-2002)

In his first publication, in 1967, Art discovered that, under appropriate conditions, oscillator populations can spontaneously synchronize after the dispersion falls below a certain threshold. The synchrony arises in a manner reminiscent of a second-order phase transition. In the years that followed, that novel approach stimulated extensive theoretical research in nonlinear dynamics and statistical mechanics, most notably by Yoshiki Kuramoto and his colleagues.

In the 1980s, Arthur Wifree pioneered the study of scroll waves, the three-dimensional counterpart of spiral waves. He found that the ends of the scrolls typically join together to form closed rings that could be diversely linked, twisted, and knotted. Those structures represent the basic particle-like solutions of the field equations for excitable media. Aside from the fundamental importance of scroll waves, Art always believed they were likely to be important in cardiac arrhythmias, especially ventricular fibrillation.

Strogatz, 2003
Winfree model (1967)

\[ \dot{\psi}_i = \omega_i + R(\psi_i) \sum_{j=1}^{N} \Gamma(\psi_j), \quad i = 1, \ldots, N \]

According to the Winfree model:
- dynamics of each phase \( \psi_i \) reacts on its own state \( \psi_i \) through the function \( R \)
- and, it is influenced by a mean-field type function \( \Gamma \).

Kuramoto model (1975)

\[ \dot{\psi}_i = \omega_i + \sum_{j=1}^{N} G(\psi_j - \psi_i), \quad i = 1, \ldots, N. \]

According to the Kuramoto model:
- dynamics of each phase \( \psi_i \) reacts only on a mean-field type function of phase differences \( G(\psi_i - \psi_i) \)
Winfree model (1967)

\[
\dot{\psi}_i = \omega_i + R(\psi_i) \sum_{j=1}^{N} \Gamma(\psi_j), \quad i = 1, \ldots, N
\]

According to the Winfree model:
- dynamics of each phase $\psi_i$ reacts on its own state $\psi_i$ through the function $R$
- and, it is influenced by a mean-field type function $\Gamma$.

‘Standard’ Kuramoto model (1975)

\[
\dot{\psi}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\psi_j - \psi_i), \quad i = 1, \ldots, N.
\]

According to the ‘standard’ Kuramoto model:
- dynamics of each phase $\psi_i$ reacts only on a mean-field type sinusoidal function of phase differences $\sin(\psi_i - \psi_i)$
Dynamics of Coupled Oscillators: 40 years of the Kuramoto Model

International Workshop
27 - 31 July 2015

40 years ago Yoshiki Kuramoto developed a solvable model describing synchronization transition in an oscillator ensemble. His seminal work initiated a broad field of research, with numerous applications ranging from physics to neuroscience.

The workshop will discuss the state of the art in understanding the dynamics of coupled oscillators, including novel concepts such as chimeras, partial synchrony, and complexity reduction. Advances in theory and experiments will be addressed by physicists, mathematicians, and applied scientists.

**Topics include**
- Dynamics of oscillator networks
- Synchronization transitions
- Phase reduction and collective variables
- Chimeras and partially synchronous states
- Neural dynamics and other applications

**Invited speakers**
(to be confirmed*)
T. Antonsen* (USA)
P. Ashwin (UK)
E. Barreto (USA)
H. Chaté (FR)
H. Chiba (JP)
P. Colet (ES)
A. Daffertshofer (NL)
H. Daido* (JP)
D. Hansel (FR)
I. Kiss (USA)
H. Kori (JP)
Y. Kuramoto* (JP)
C. Laing* (NZ)
Y. Maistrenko (UA)
E. Martens (DK)
R. Mirollo (USA)
H. Nakao (JP)
O. Omel’chenko (DE)
E. Ott (USA)
D. Pazó (ES)
J. Restrepo (USA)
S. Ruffo (IT)
L. Schimansky-Geier (DE)
K. Showalter (USA)
P. So (USA)
A. Stefanovska (UK)
S. Strogatz (USA)
M. Timme (DE)
A. Torcini (IT)
K. Wiesenfeld* (USA)

**Scientific coordinators**
Arkady Pikovsky
Potsdam, DE

Antonio Politi
Aberdeen, UK

Michael Rosenblum
Potsdam, DE

**Organisation**
Maria Pätzold, MIPK
Kuramoto talks about the Kuramoto model  (July 2015)

https://www.youtube.com/watch?v=lac4TxWyBOg
1:00 – 13:00 min.

Strogatz talks about the Kuramoto model  (July 2015)

https://www.youtube.com/watch?v=Zto3Q0mS_Nw
3:00 – 9:00  min
Kuramoto model (1975)

Network of \( N \) coupled phase oscillators:

\[
\frac{d\psi_i}{dt} = \omega_i + \sum_{j=1}^{N} G_{ij}(\psi_j - \psi_i), \quad i = 1, \ldots, N
\]

- \( \psi_i \): phases of individual oscillators
- \( \omega_i \): frequencies of individual oscillators are distributed according to some probabilistic density \( g(\omega) \), like Gaussian distribution.
- \( G_{ij}(\psi_j - \psi_i) \): coupling function
Kuramoto model (1975)

Network of $N$ coupled phase oscillators:

$$\frac{d\psi_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\psi_j - \psi_i), \quad i = 1, \ldots, N$$

$\psi_i$ - phases of individual oscillators

$\omega_i$ - frequencies of individual oscillators are distributed according to some probabilistic density $g(\omega)$, like Gaussian distribution.

$\sin(\psi_j - \psi_i)$ - coupling function
How one can get Kuramoto model?

One-dimensional complex Ginzburg-Landau equation, periodic boundary conditions

\[
\frac{\partial}{\partial t} A(x, t) = (1 + i\omega_0)A - (1 + ib)|A|^2 A + (1 + i\alpha) \frac{\partial^2 A}{\partial x^2}
\]

Introduce integral coupling

\[
\frac{\partial}{\partial t} A(x, t) = (1 + i\omega_0)A - (1 + ib)|A|^2 A + K(1 + ia) \left( Z(x, t) - A(x, t) \right)
\]

Integral coupling term

\[
Z(x, t) = \int G(x - x') A(x', t) dx'
\]

Integral Kuramoto model

\[
\frac{\partial \phi}{\partial t} = \omega - \int_{-\pi}^{\pi} G(x - x') \sin[\phi(x, t) - \phi(x', t) + \alpha] dx'
\]

Discrete Kuramoto model

\[
\dot{\phi}_i = \omega + \frac{1}{N} \sum_{j=1}^{N} G_{ij} \sin(\phi_j - \phi_i - \alpha)
\]

Parameter \(\alpha\)

\[
\tan \alpha = \frac{b - a}{1 + ab}, \quad \alpha(b - a) > 0
\]
‘Standard’ Kuramoto model

All-to-all sinusoidal coupling: \( G_{ij}(\phi) = \frac{K}{N} \sin \phi \) \hspace{1cm} (\phi \equiv \psi_j - \psi_i)

\[ \dot{\psi}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\psi_j - \psi_i), \quad i = 1,\ldots, N. \]

There exists a critical bifurcation value \( K_c > 0 \):

\( K > K_c \) : synchronization (phase-locking)
\( K < K_c \) : desynchronization (clustering and chaos)

The Kuramoto transition:

When the coupling is small compared to the spread of natural frequencies, the system behaves incoherently, with each oscillator running at its natural frequency. As the coupling is increased, the incoherence persists until a certain threshold \( K = K_c \) is crossed — then a small cluster of oscillators suddenly ‘freezes’ into synchrony. For still greater coupling, all the oscillators become locked in phase and amplitude.
Synchronization transition in the Kuramoto model

\[ N = 3 \quad N = 7 \]

\[ \bar{\omega}_i = \bar{\omega}_2 \]

\[ \bar{\omega}_3 \]

\[ K_{c1} \quad 0.4 \quad 0.6 \quad K_c \quad 0.8 \quad 1.0 \]

\[ K \]

\[ \bar{\omega}_i \quad - \text{average frequency of individual oscillator (Poincare rotation number) } \]

\[ \bar{\omega}_i = \lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{\psi}(t) dt \]
Nil, partial and full phase-locking in an all-to-all network of Kuramoto oscillators. Phase-locking is governed by the coupling strength $K$ and the distribution of intrinsic frequencies $\omega$. Here, the intrinsic frequencies were drawn from a normal distribution ($M=0.5\text{Hz}, SD=0.5\text{Hz}$). The yellow disk marks the phase centroid. Its radius is a measure of coherence.

Video:
http://www.youtube.com/watch?v=ZYdaZO9odNc
http://www.youtube.com/watch?v=Ujk22Flv2mE
Complex order parameter

\[ \psi_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\psi_j - \psi_i), \quad i = 1, \ldots, N. \]

ORDER PARAMETER:

\[ re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\psi_j} \]

\( r(t) \) measures the coherence of the oscillator population
\( \psi(t) \) is the average phase

\[ \dot{\psi}_i = \omega_i + Kr \sin(\psi - \psi_i), \quad i = 1, \ldots, N. \]

Each oscillator \( \psi_i \) is coupled to the common average phase \( \psi(t) \) with coupling strength given by \( Kr \)

KuramotoModelPhaseLocking.ogv.360p.webm
Cauchy-Lorentz frequency distribution \((N = \infty)\)

\[
r = \sqrt{1 - \frac{K_c}{K}} \quad \text{for} \quad g(\omega) = \frac{\gamma}{\pi (\gamma^2 + \omega^2)}
\]

“Second order phase transition”

Near onset

\[
r \approx \sqrt{\frac{16}{\pi K_c^3}} \sqrt{\frac{\mu}{-g''(0)}}
\]

Supercritical bifurcation for \(g''(0) < 0\)

\[
K_c = \frac{2}{\pi g(0)}
\]
Two simple properties of Kuramoto model

\[ \dot{\psi}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\psi_j - \psi_i), \quad i = 1, \ldots, N. \]

**Th. 1** Preserving mean frequency:

\[ \Omega(t) = \frac{1}{N} \sum_{j=1}^{N} \dot{\psi}_i(t) = \text{Const} \quad (= \frac{1}{N} \sum_{j=1}^{N} \omega_i) \]

**Th. 2** Reduction from \( N \)-Dim to \( (N - 1) \)-Dim system in phase differences

\[ \phi_i = \psi_{i+1} - \psi_i, \quad i = 1, \ldots, N - 1 \]
\[ \dot{\psi}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin (\psi_j - \psi_i), \quad i = 1, \ldots, N \quad (1) \]

(All-to-all coupling, no topology)
\[ \omega_i \] - natural frequency of oscillator \( \psi_i \)

Def. \[ S_0 = \frac{1}{N} \sum_{i=1}^{N} \omega_i \] - mean natural frequency

Th. 1
\[ \frac{1}{N} \sum_{i=1}^{N} \dot{\psi}_i (t) = S_0 \] (Preserving mean frequency)

Proof: just to sum

Corollary
\[ \frac{1}{N} \sum_{i=1}^{N} \psi_i (t) = S_0 t + C \] (first integral of system (1))

\[ \Rightarrow \text{mean phase } \psi = \frac{1}{N} \sum_{i=1}^{N} \psi_i \text{ rotates with constant velocity } S_0 \]

Change variables
\[ \psi_i = \tilde{\psi}_i + S_0 t \]

mean velocity \( \Rightarrow 0 \)

mean phase \( \Rightarrow C \)

\[ \dot{\tilde{\psi}}_i + S_0 = \dot{\psi}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin (\tilde{\psi}_j - \tilde{\psi}_i) \]

\[ \dot{\tilde{\psi}}_i = \tilde{\psi}_i + \frac{K}{N} \sum_{j=1}^{N} \sin (\tilde{\psi}_j - \tilde{\psi}_i) \]

\[ \frac{1}{N} \sum_{i=1}^{N} \dot{\psi}_i (t) = \frac{1}{N} \sum_{i=1}^{N} \tilde{\psi}_i = 0 \]

\[ \Rightarrow \quad \text{mean phase } \psi = 0 \quad \forall t \geq 0 \]
\[
\dot{\psi}_i = \Delta_i + \frac{K}{N} \left[ \sum_{j=1}^{N-1} \sin(\psi_j - \psi_i) - \sum_{j=1}^{N-1} \sin \psi_j + \sin \psi_i \right] \quad (i = 1, \ldots, N-1)
\]

**Corollary**

Kuramoto model has \((N-1)\) Dim dynamics (i.e. at \((N-1)\) Dim manifold in \(N\) Dim space)
The End