1 Regular perturbation expansion for dynamical systems

1.1 Setting

Good introductory texts into the topic of perturbation theory are [1, 2, 4, 5]. Consider the n-component *state vector* $\boldsymbol{x}(t) \in \mathbb{R}^n$,

$$\boldsymbol{x}(t) = (x_1(t), \dots, x_n(t))^T, \tag{1}$$

and a typically nonlinear function $\boldsymbol{R}(\boldsymbol{x})$

$$\boldsymbol{R}(\boldsymbol{x}) = (R_1(\boldsymbol{x}), \dots, R_n(\boldsymbol{x}))^T. \tag{2}$$

A *dynamical system* is a system of ordinary differential equations (ODE) given by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{3}$$

with $\dot{\boldsymbol{x}}(t)$ denoting the time derivative and $\boldsymbol{x}_0$ the *initial condition*.
The *perturbed dynamical system* is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \epsilon \boldsymbol{f}(\boldsymbol{x}(t), t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 + \epsilon \boldsymbol{x}_1. \tag{4}$$

We distinguish two types of perturbations:

1. perturbations $\boldsymbol{x}_1$ of the initial conditions and
2. structural perturbations $\boldsymbol{f}(\boldsymbol{x}(t), t)$ of the dynamical system itself. These perturbations may depend on the state $\boldsymbol{x}$ and explicitly on time t.

To be able to apply a perturbation expansion, the perturbations are assumed to be small. This is indicated by the small coefficient $0 \leq \epsilon \ll 1$. Note that $\boldsymbol{x}_1$ and $\boldsymbol{f}$ are assumed to be independent of ϵ. A more general perturbed dynamical system is given by

$$\dot{\boldsymbol{x}}(t) = \tilde{\boldsymbol{R}}(\boldsymbol{x}(t)) + \tilde{\boldsymbol{f}}(\boldsymbol{x}(t), t; \epsilon), \qquad \boldsymbol{x}(t_0) = \tilde{\boldsymbol{x}}_0(\epsilon). \tag{5}$$

For a perturbative treatment of Eq. (5), $\tilde{\boldsymbol{f}}$ as well as $\tilde{\boldsymbol{x}}_0$ are expanded in a Taylor series in ϵ.

1.2 Taylor expansion

We expand the perturbed state $\boldsymbol{x}(t) = \boldsymbol{X}(t; \epsilon)$ in a Taylor series in ϵ as

$$\begin{aligned} \boldsymbol{X}(t; \epsilon) &= \sum_{k=0}^{\infty} \frac{1}{k!} \partial_\epsilon \boldsymbol{X}(t; \epsilon) \bigg|_{\epsilon=0} \\ &= \boldsymbol{X}_0(t) + \epsilon \boldsymbol{X}_1(t) + \frac{\epsilon^2}{2} \boldsymbol{X}_2(t) + \mathcal{O}(\epsilon^3). \end{aligned} \tag{6}$$

The notation $\mathcal{O}(\epsilon^\alpha)$ means that the terms which are neglected in the series expansion are of order equal to or larger than ϵ^α. Using Eq. (6) in Eqs. (4) yields

$$\begin{aligned}\dot{\boldsymbol{X}}_0(t)+\epsilon\dot{\boldsymbol{X}}_1(t)+\mathcal{O}\left(\epsilon^2\right)&=\boldsymbol{R}\left(\boldsymbol{X}_0(t)+\epsilon\boldsymbol{X}_1(t)+\mathcal{O}\left(\epsilon^2\right)\right)\\&\quad+\epsilon\boldsymbol{f}\left(\boldsymbol{X}_0(t)+\epsilon\boldsymbol{X}_1(t)+\mathcal{O}\left(\epsilon^2\right),t\right),\end{aligned}\tag{7}$$

$$\boldsymbol{X}_0(t_0)+\epsilon\boldsymbol{X}_1(t_0)+\mathcal{O}\left(\epsilon^2\right)=\boldsymbol{x}_0+\epsilon\boldsymbol{x}_1.\tag{8}$$

Expanding $\boldsymbol{R}$ in a Taylor series in ϵ yields

$$\begin{aligned}&\boldsymbol{R}\left(\boldsymbol{X}_0(t)+\epsilon\boldsymbol{X}_1(t)+\epsilon^2\boldsymbol{X}_2(t)+\mathcal{O}\left(\epsilon^3\right)\right)=\boldsymbol{R}\left(\boldsymbol{X}_0(t)\right)\\&+\epsilon\nabla_{\boldsymbol{x}}\boldsymbol{R}\left(\boldsymbol{X}_0(t)\right)\boldsymbol{X}_1(t)+\frac{\epsilon^2}{2}\boldsymbol{X}_1^T(t)\nabla_{\boldsymbol{x}}^2\boldsymbol{R}\left(\boldsymbol{X}_0(t)\right)\boldsymbol{X}_1(t)+\mathcal{O}\left(\epsilon^3\right).\end{aligned}\tag{9}$$

Here,

$$\boldsymbol{\mathcal{A}}(t)=\nabla_{\boldsymbol{x}}\boldsymbol{R}\left(\boldsymbol{X}_0(t)\right)\tag{10}$$

denotes the Jacobian matrix of $\boldsymbol{R}(\boldsymbol{x})$ evaluated at the solution $\boldsymbol{x}=\boldsymbol{X}_0(t)$ to the unperturbed dynamical system (3). The components of $\boldsymbol{\mathcal{A}}(t)$ are given by

$$\mathcal{A}_{ij}(t)=\frac{\partial}{\partial x_j}R_i(\boldsymbol{x})\bigg|_{\boldsymbol{x}=\boldsymbol{X}_0(t)}.\tag{11}$$

Similarly, $\nabla_{\boldsymbol{x}}^2\boldsymbol{R}\left(\boldsymbol{X}_0(t)\right)$ denotes the third order tensor involving all second order derivatives of $\boldsymbol{R}(\boldsymbol{x})$ with respect to the components of $\boldsymbol{x}$ evaluated at $\boldsymbol{x}=\boldsymbol{X}_0(t)$. In components, the relevant term is

$$\left(\boldsymbol{X}_1^T(t)\nabla_{\boldsymbol{x}}^2\boldsymbol{R}\left(\boldsymbol{X}_0(t)\right)\boldsymbol{X}_1(t)\right)_i=\sum_{j,k=1}^{n}X_{1,j}(t)\,X_{1,k}(t)\,\frac{\partial^2}{\partial x_j\partial x_k}R_i(\boldsymbol{x})\bigg|_{\boldsymbol{x}=\boldsymbol{X}_0(t)},\tag{12}$$

and $\boldsymbol{X}_1(t)=\left(X_{1,1}(t),\dots,X_{1,n}(t)\right)^T$. Finally, collecting terms of equal order of ϵ in Eq. (7), we obtain the following set of equations. The equation to leading order of ϵ, denoted by $\mathcal{O}(1)$, are

$$\mathcal{O}(1):\qquad\dot{\boldsymbol{X}}_0(t)=\boldsymbol{R}\left(\boldsymbol{X}_0(t)\right),\qquad\boldsymbol{X}_0(t_0)=\boldsymbol{x}_0.\tag{13}$$

This nonlinear ordinary differential equation is identical to the unperturbed dynamical system (3). In first order of ϵ, we obtain

$$\mathcal{O}(\epsilon):\qquad\dot{\boldsymbol{X}}_1(t)=\boldsymbol{\mathcal{A}}(t)\,\boldsymbol{X}_1(t)+\boldsymbol{f}\left(\boldsymbol{X}_0(t),t\right),\boldsymbol{X}_1(t_0)=\boldsymbol{x}_1.\tag{14}$$

This is an inhomogeneous linear ordinary differential equation with a state matrix $\boldsymbol{\mathcal{A}}(t)$ given by Eq. The inhomogeneity $\boldsymbol{f}$ depends explicitly on time t, and implicitly through $\boldsymbol{X}_0(t)$. Since Eq. (14) is a linear ODE, its formal solution can be readily given in terms of the state transition matrix $\boldsymbol{\Phi}(t, t_0)$ as

$$\boldsymbol{X}_1(t) = \boldsymbol{\Phi}(t, t_0)\, \boldsymbol{x}_1 + \int_{t_0}^{t} \mathrm{d}\tau \boldsymbol{\Phi}(t, \tau)\, \boldsymbol{f}(\boldsymbol{X}_0(\tau), \tau). \tag{15}$$

The state transition matrix satisfies

$$\partial_t \boldsymbol{\Phi}(t, \tau) = \boldsymbol{\mathcal{A}}(t)\, \boldsymbol{\Phi}(t, \tau), \qquad \boldsymbol{\Phi}(\tau, \tau) = \mathbf{1}. \tag{16}$$

For a constant state matrix

$$\boldsymbol{\mathcal{A}}(t) = \boldsymbol{\mathcal{A}} = \text{const.}, \tag{17}$$

the solution for $\boldsymbol{\Phi}$ is given in terms of the matrix exponential

$$\boldsymbol{\Phi}(t, \tau) = \exp\left(\boldsymbol{\mathcal{A}}(t - \tau)\right) = \sum_{k=0}^{\infty} \frac{\boldsymbol{\mathcal{A}}^k}{k!} (t - \tau)^k. \tag{18}$$

A proof of Eq. (18) is simple:

$$\begin{aligned}\partial_t \boldsymbol{\Phi}(t, \tau) &= \sum_{k=0}^{\infty} \frac{\boldsymbol{\mathcal{A}}^k}{k!} \partial_t (t - \tau)^k = \sum_{k=1}^{\infty} \boldsymbol{\mathcal{A}}^k \frac{k}{k!} (t - \tau)^{k-1} \\ &= \boldsymbol{\mathcal{A}} \sum_{k=1}^{\infty} \frac{\boldsymbol{\mathcal{A}}^{k-1}}{(k-1)!} (t - \tau)^{k-1} = \boldsymbol{\mathcal{A}} \sum_{\tilde{k}=0}^{\infty} \frac{\boldsymbol{\mathcal{A}}^{\tilde{k}}}{\tilde{k}!} (t - \tau)^{\tilde{k}} = \boldsymbol{\mathcal{A}} \boldsymbol{\Phi}(t, \tau),\end{aligned} \tag{19}$$

where we introduced the index shift $\tilde{k} = k - 1$ in the second line. If the matrix $\boldsymbol{\mathcal{A}}$ can be diagonalized, there exists an invertible matrix $\boldsymbol{\mathcal{P}}$ such that

$$\boldsymbol{\mathcal{P}}^{-1} \boldsymbol{\mathcal{A}} \boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{A}}_D \tag{20}$$

and $\boldsymbol{\mathcal{A}}_D$ has the eigenvalues λ_i, $i \in \{1, \dots, n\}$ of $\boldsymbol{\mathcal{A}}$ on its diagonal. The matrix exponential becomes

$$\begin{aligned}\exp\left(\boldsymbol{\mathcal{A}}(t - \tau)\right) &= \sum_{k=0}^{\infty} \frac{\boldsymbol{\mathcal{A}}^k}{k!} (t - \tau)^k = \sum_{k=0}^{\infty} \frac{\left(\boldsymbol{\mathcal{P}} \boldsymbol{\mathcal{A}}_D \boldsymbol{\mathcal{P}}^{-1}\right)^k}{k!} (t - \tau)^k \\ &= \boldsymbol{\mathcal{P}} \sum_{k=0}^{\infty} \frac{\boldsymbol{\mathcal{A}}_D^k}{k!} (t - \tau)^k \boldsymbol{\mathcal{P}}^{-1} = \boldsymbol{\mathcal{P}} \exp\left(\boldsymbol{\mathcal{A}}_D (t - \tau)\right) \boldsymbol{\mathcal{P}}^{-1}.\end{aligned} \tag{21}$$

Because $\boldsymbol{\mathcal{A}}_D$ is diagonal, its matrix exponential is diagonal as well with

$$\exp\left(\boldsymbol{\mathcal{A}}_D (t - \tau)\right) = \begin{pmatrix} \exp\left(\lambda_1 (t - \tau)\right) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \exp\left(\lambda_n (t - \tau)\right) \end{pmatrix}. \tag{22}$$

Consider the perturbed dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)), \qquad\qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 + \epsilon \boldsymbol{x}_1, \tag{23}$$

and assume that $\boldsymbol{x}_0$ is a stationary point of the unperturbed dynamical system, i.e,

$$\boldsymbol{R}(\boldsymbol{x}_0) = \mathbf{0}. \tag{24}$$

The perturbed solution is

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \epsilon \boldsymbol{X}_1(t) + \mathcal{O}\left(\epsilon^2\right), \tag{25}$$

with $\boldsymbol{X}_1(t)$ given according to Eqs. (15) and (21) as

$$\boldsymbol{X}_1(t) = \boldsymbol{\mathcal{P}} \exp\left(\boldsymbol{\mathcal{A}}_D(t-\tau)\right) \boldsymbol{\mathcal{P}}^{-1} \boldsymbol{x}_1. \tag{26}$$

The asymptotic behavior of $\boldsymbol{x}(t)$ for large times can be read off Eq. (26) together with Eq. (22). If the real parts of all eigenvalues of $\boldsymbol{\mathcal{A}}$ are strictly negative, $\Re(\lambda_i) < 0$ for all $i \in \{1, \ldots, n\}$, the solution $\boldsymbol{x}(t)$ approaches the stationary point $\boldsymbol{x}_0$ for large times,

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \lim_{t\to\infty} \left(\boldsymbol{x}_0 + \epsilon \boldsymbol{X}_1(t)\right) = \boldsymbol{x}_0. \tag{27}$$

Thus, if $\Re(\lambda_i) < 0$ for all $i \in \{1, \ldots, n\}$, a small perturbation of the initial conditions decays for large times, and the stationary point $\boldsymbol{x}_0$ is considered stable. If $\Re(\lambda_i) > 0$ for one $i \in \{1, \ldots, n\}$ the stationary point $\boldsymbol{x}_0$ is unstable. The situation with $\Re(\lambda_i) = 0$ for one $i \in \{1, \ldots, n\}$ is called marginally stable. To draw definite conclusions about the stability of $\boldsymbol{x}_0$ in this case, the perturbation expansion must be carried out to higher orders of ϵ.
For general time-dependent state matrices $\boldsymbol{\mathcal{A}}(t)$, no simple expression for the state transition matrix $\boldsymbol{\Phi}$ exists. Furthermore, the time-dependent eigenvalues $\lambda_i(t)$ of $\boldsymbol{\mathcal{A}}(t)$ do not tell us anything about the asymptotic behavior of $\boldsymbol{X}_1(t)$ for large times. To determine the stability of time-dependent unperturbed solutions $\boldsymbol{X}_0(t)$ requires different techniques as e.g. Lyapunov functions.
In next higher order of ϵ, the equation for $\boldsymbol{X}_2(t)$ read as

$$\begin{aligned} \mathcal{O}\left(\epsilon^2\right): \qquad \dot{\boldsymbol{X}}_2(t) &= \nabla_{\boldsymbol{x}} \boldsymbol{R}(\boldsymbol{X}_0(t)) \boldsymbol{X}_2(t) + \boldsymbol{X}_1^T(t) \nabla_{\boldsymbol{x}}^2 \boldsymbol{R}(\boldsymbol{X}_0(t)) \boldsymbol{X}_1(t) \\ &\quad + 2\nabla_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{X}_0(t), t) \boldsymbol{X}_1(t), \qquad (28) \\ \boldsymbol{X}_2(t_0) &= \mathbf{0}. \qquad (29) \end{aligned}$$

Note that again we obtained a linear inhomogeneous differential equation. This will be true to any order of ϵ^n with $n \geq 1$. The only nonlinear differential equation is the unperturbed dynamical system Eq. (3). If an exact solution $\boldsymbol{X}_0(t)$ is analytically known, all higher order contributions to the solution $\boldsymbol{x}(t)$ of the perturbed dynamical system (4) are obtained by solving only linear inhomogeneous differential equations.

Example. *Stability of stationary points in the plane.*
Consider the two-dimensional dynamical system

$$\dot{x}(t) = f(x(t), y(t)), \qquad \dot{y}(t) = g(x(t), y(t)), \tag{30}$$

$$x(t_0) = x_0 + \epsilon x_1, \qquad y(t_0) = y_0 + \epsilon y_1. \tag{31}$$

The initial condition $(x_0, y_0)^T$ is assumed to be a stationary solution of Eq. (30), i.e.

$$f(x_0, y_0) = g(x_0, y_0) = 0. \tag{32}$$

The linear system arising in first order perturbation theory in the small parameter ϵ,

$$\dot{\boldsymbol{X}}_1(t) = \boldsymbol{\mathcal{A}}\boldsymbol{X}_1(t), \qquad \boldsymbol{X}_1(t_0) = (x_1, y_1)^T, \tag{33}$$

is characterized by the 2×2 matrix

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}, \tag{34}$$

with $f_x(x_0, y_0)$ denoting the derivative of $f(x, y)$ with respect to x evaluated at the stationary point $(x_0, y_0)^T$. The stability of the stationary point is determined by the eigenvalues of $\boldsymbol{\mathcal{A}}$ given by

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}(a_{11} + a_{22}) \pm \frac{1}{2}\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}} \\ &= \frac{1}{2}\mathrm{tr}(\boldsymbol{\mathcal{A}}) \pm \frac{1}{2}\sqrt{\mathrm{tr}(\boldsymbol{\mathcal{A}})^2 - 4\det(\boldsymbol{\mathcal{A}})}. \end{aligned} \tag{35}$$

Here, $\mathrm{tr}(\boldsymbol{\mathcal{A}})$ denotes the trace of $\boldsymbol{\mathcal{A}}$ and $\det(\boldsymbol{\mathcal{A}})$ denotes the determinant of $\boldsymbol{\mathcal{A}}$. A phase portrait displaying the possible behaviors of $\boldsymbol{X}_1(t) = (x(t), y(t))^T$ in the (x, y) phase plane in dependence of $\mathrm{tr}(\boldsymbol{\mathcal{A}})$ and $\det(\boldsymbol{\mathcal{A}})$ is shown in Fig. 1. For an extensive discussion of all possible cases see the appendix of [3].

1.3 Asymptotic series

In section 1.2, we performed a series expansion of a dynamical system in form of a Taylor series in the small parameter ϵ around $\epsilon = 0$. An expansion in form of a Taylor series is an assumption which is not always justified. This is best understood with the help of an example. Consider the function

$$g(\epsilon) = \sqrt{1 - \frac{1}{1 + \epsilon}}. \tag{36}$$

The first two terms of a Taylor expansion of $g(\epsilon)$ would be

$$g(\epsilon) = 0 + \epsilon g'(0) + \mathcal{O}(\epsilon^2). \tag{37}$$

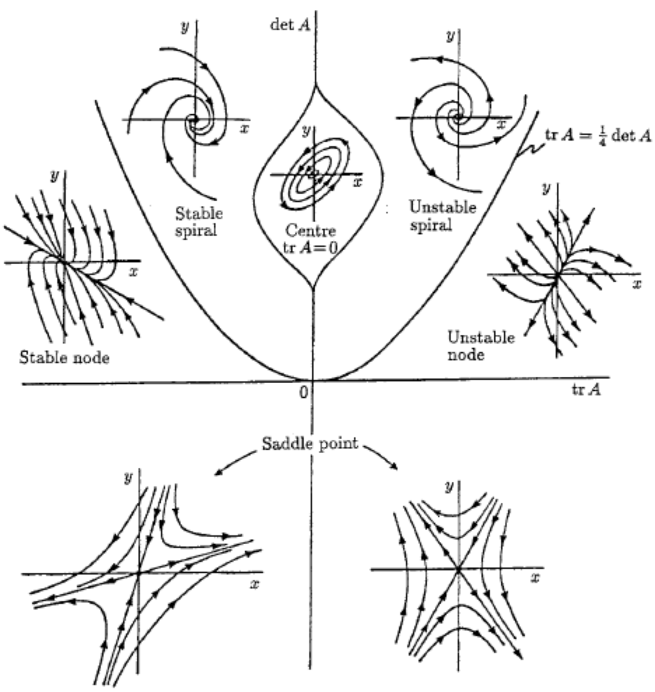

Figure 1: Possible behavior of $\boldsymbol{X}_1(t) = (x(t), y(t))^T$ given as the solution to the linear homogeneous ODE Eq. (33) in dependence of $\mathrm{tr}(\boldsymbol{\mathcal{A}})$ and $\det(\boldsymbol{\mathcal{A}})$. From [3].

However, the derivative of $g(\epsilon)$ is

$$g'(\epsilon) = \frac{1}{2\sqrt{\epsilon}(1+\epsilon)^{3/2}} \tag{38}$$

which diverges for $\epsilon \to 0$. Thus, $g(\epsilon)$ cannot be expanded in a Taylor series in ϵ. Nevertheless, an alternative series expansion is possible:

$$g(\epsilon) = \sqrt{\epsilon} - \frac{\epsilon^{3/2}}{2} + \frac{3\epsilon^{5/2}}{8} + \mathcal{O}\left(\epsilon^{7/2}\right). \tag{39}$$

Generally, an expansion of the form

$$g(\epsilon) = \sum_{k=1}^{N} a_k \varphi_k(\epsilon), \tag{40}$$

with

$$\lim_{\epsilon \to 0} \frac{\varphi_{k+1}(\epsilon)}{\varphi_k(\epsilon)} = 0, \tag{41}$$

is known as an asymptotic series. Examples for sequences of functions $\varphi_k(\epsilon)$ are

$$\varphi_1(\epsilon) = \epsilon^{1/2}, \qquad \varphi_2(\epsilon) = \epsilon^1, \qquad \varphi_3(\epsilon) = \epsilon^{3/2}, \dots, \tag{42}$$

and

$$\varphi_1(\epsilon) = \ln(\epsilon), \quad \varphi_2(\epsilon) = \ln(\ln(\epsilon)), \quad \varphi_3(\epsilon) = \ln(\ln(\ln(\epsilon))), \dots. \tag{43}$$

How is the asymptotic series for $g(\epsilon)$ determined? In particular, how do we calculate the exponents $\alpha_1, \alpha_2, \dots$ and coefficients $a_1, a_2, \dots$ of $a_i\epsilon^{\alpha_i}$ in Eq. (39)? For simplicity, we use the ansatz

$$g(\epsilon) = a_1\epsilon^{\alpha_1} + a_2\epsilon^{\alpha_2} + \dots \tag{44}$$

with $0 < \alpha_1 < \alpha_2$. To determine α_1 and a_1, we can write

$$\frac{g(\epsilon)}{\epsilon^{\alpha_1}} = a_1 + a_2\epsilon^{\alpha_2 - \alpha_1} + \dots. \tag{45}$$

Taking the limit $\epsilon \to 0$ of Eq. (45) yields, depending on α_1, three possibilities. For most values of α_1, the limit may diverge, i.e., it does not exist, or it vanishes. However, for one particular value of α_1, the limit yields a finite non-vanishing value. In this case, α_1 is said to satisfy a dominant balance. We obtain the value for a_1 as

$$\lim_{\epsilon \to 0} \frac{g(\epsilon)}{\epsilon^{\alpha_1}} = a_1. \tag{46}$$

To determine α_1, we compute

$$\frac{\sqrt{1-\frac{1}{1+\epsilon}}}{\epsilon^{\alpha_1}} = \frac{\sqrt{1-1+\epsilon+\mathcal{O}\left(\epsilon^2\right)}}{\epsilon^{\alpha_1}} = \sqrt{\epsilon^{1-2\alpha_1}+\mathcal{O}\left(\epsilon^{2(1-\alpha_1)}\right)} \tag{47}$$

We used $(1+\epsilon)^{-1} = 1-\epsilon+\mathcal{O}\left(\epsilon^2\right)$ and $\epsilon^{\alpha}\mathcal{O}\left(\epsilon^{\beta}\right) = \mathcal{O}\left(\epsilon^{\alpha+\beta}\right)$. From Eq. (47), it is obvious that only the value $\alpha_1 = 1/2$ yields a finite value in the limit $\epsilon \to 0$. Thus, we may compute the coefficient a_1 as

$$\begin{aligned} a_1 &= \lim_{\epsilon\to 0}\frac{g(\epsilon)}{\epsilon^{1/2}} = \frac{0}{0} = \lim_{\epsilon\to 0}\frac{g'(\epsilon)}{\frac{1}{2}\epsilon^{-1/2}} \\ &= \lim_{\epsilon\to 0}\frac{2}{2\epsilon^{1/2}(1+\epsilon)^{3/2}}\epsilon^{1/2} = \lim_{\epsilon\to 0}\frac{1}{(1+\epsilon)^{3/2}} = 1, \end{aligned} \tag{48}$$

where we used L'Hôspital's rule in the first line. To obtain the coefficient and exponent a_2 and α_2, we compute

$$\begin{aligned} a_2 &= \lim_{\epsilon\to 0}\frac{g(\epsilon)-a_1\epsilon^{\alpha_1}}{\epsilon^{\alpha_2}} = \lim_{\epsilon\to 0}\frac{\sqrt{1-\frac{1}{1+\epsilon}}-\epsilon^{1/2}}{\epsilon^{\alpha_2}} \\ &= \lim_{\epsilon\to 0}\frac{\epsilon^{1/2}\left(\sqrt{1-\epsilon+\mathcal{O}\left(\epsilon^2\right)}-1\right)}{\epsilon^{\alpha_2}} = \lim_{\epsilon\to 0}\frac{-\frac{\epsilon^{3/2}}{2}+\mathcal{O}\left(\epsilon^{5/2}\right)}{\epsilon^{\alpha_2}} = -\frac{1}{2}. \end{aligned} \tag{49}$$

We used $(1+\epsilon)^{-1} = 1-\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)$ in the first and $\sqrt{1-\epsilon} = 1-\frac{\epsilon}{2}+\mathcal{O}\left(\epsilon^2\right)$ in the second line. The exponent α_2 is determined as $\alpha_2 = 3/2$ to obtain a dominant balance. Finally, the expansion of $g(\epsilon)$ in form of an asymptotic series reads as

$$g(\epsilon) = \epsilon^{1/2} - \frac{\epsilon^{3/2}}{2} + \dots. \tag{50}$$

2 Regular perturbation expansion for reaction-diffusion systems

Consider the perturbed reaction-diffusion system for the state $\boldsymbol{x}(\boldsymbol{r},t) \in \mathbb{R}^n$,

$$\partial_t \boldsymbol{x}(\boldsymbol{r},t) = \boldsymbol{\mathcal{D}}\triangle_{\boldsymbol{r}}\boldsymbol{x}(\boldsymbol{r},t) + \boldsymbol{R}(\boldsymbol{x}(\boldsymbol{r},t)) + \epsilon\boldsymbol{f}(\boldsymbol{x}(\boldsymbol{r},t),\boldsymbol{r},t), \tag{51}$$

$$\boldsymbol{x}(\boldsymbol{r},t_0) = \boldsymbol{x}_0(\boldsymbol{r}) + \epsilon\boldsymbol{x}_1(\boldsymbol{r}). \tag{52}$$

For simplicity, we consider an infinite spatial domain $\boldsymbol{r} = (r_1,\dots,r_N)^T \in \mathbb{R}^N$ of arbitrary dimension N. The Laplacian $\triangle$ acts component-wise on $\boldsymbol{x}(\boldsymbol{r},t)$. In Cartesian coordinates, it reads as

$$\triangle_{\boldsymbol{r}} = \sum_{i=1}^{N}\frac{\partial^2}{\partial r_i^2}. \tag{53}$$

The matrix of diffusion coefficients $\boldsymbol{\mathcal{D}}$ is assumed to be diagonal,

$$\mathcal{D}_{ij} = D_i \delta_{ij} \tag{54}$$

with Kronecker delta δ_{ij} such that the medium is isotropic.
For the regular perturbation expansion we use an ansatz in form of a Taylor series,

$$\boldsymbol{X}(\boldsymbol{r}, t; \epsilon) = \boldsymbol{X}_0(t) + \epsilon \boldsymbol{X}_1(t) + \mathcal{O}\left(\epsilon^2\right). \tag{55}$$

The unperturbed system obtained in $\mathcal{O}(1)$ is

$$\partial_t \boldsymbol{X}_0(\boldsymbol{r}, t) = \boldsymbol{\mathcal{D}} \triangle_{\boldsymbol{r}} \boldsymbol{X}_0(\boldsymbol{r}, t) + \boldsymbol{R}(\boldsymbol{X}_0(\boldsymbol{r}, t)), \tag{56}$$

$$\boldsymbol{X}_0(\boldsymbol{r}, t_0) = \boldsymbol{x}_0(\boldsymbol{r}). \tag{57}$$

In $\mathcal{O}(\epsilon)$, we get a linear inhomogeneous partial differential equation

$$\partial_t \boldsymbol{X}_1(\boldsymbol{r}, t) = \boldsymbol{\mathcal{D}} \triangle_{\boldsymbol{r}} \boldsymbol{X}_1(\boldsymbol{r}, t) + \nabla_{\boldsymbol{x}} \boldsymbol{R}(\boldsymbol{X}_0(\boldsymbol{r}, t)) \boldsymbol{X}_1(\boldsymbol{r}, t) + \boldsymbol{f}(\boldsymbol{X}_0(\boldsymbol{r}, t), \boldsymbol{r}, t), \tag{58}$$

$$\boldsymbol{X}_1(\boldsymbol{r}, t_0) = \boldsymbol{x}_1(\boldsymbol{r}), \tag{59}$$

with $\nabla \boldsymbol{R}(\boldsymbol{X}_0(\boldsymbol{r}, t))$ the Jacobian of $\boldsymbol{R}$ evaluated at the unperturbed solution $\boldsymbol{X}_0(\boldsymbol{r}, t)$.
For simplicity, we consider a stationary and homogeneous solution

$$\boldsymbol{X}_0(\boldsymbol{r}, t) = \boldsymbol{X}_0 \tag{60}$$

for the unperturbed system Eq. (56) in the following. To solve Eq. (58), we introduce the Fourier transform $\tilde{\boldsymbol{x}}(\boldsymbol{k}, t)$, with the vector of wave numbers $\boldsymbol{k} \in \mathbb{R}^N$, of a function $\boldsymbol{x}(\boldsymbol{r}, t)$ with respect to space $\boldsymbol{r}$,

$$\tilde{\boldsymbol{x}}(\boldsymbol{k}, t) = \hat{F}[\boldsymbol{x}(\boldsymbol{r}, t)] = \frac{1}{(2\pi)^{N/2}} \int\limits_{-\infty}^{\infty} \mathrm{d}^N \boldsymbol{r} \exp(-\imath \boldsymbol{k} \cdot \boldsymbol{r}) \boldsymbol{x}(\boldsymbol{r}, t). \tag{61}$$

Applying the Fourier transform to $\triangle_{\boldsymbol{r}} \boldsymbol{x}(\boldsymbol{r}, t)$ yields, after two-fold partial integration together with the assumption of vanishing boundary terms and $k^2 = \boldsymbol{k} \cdot \boldsymbol{k}$,

$$\hat{F}[\triangle_{\boldsymbol{r}} \boldsymbol{x}(\boldsymbol{r}, t)] = -\frac{1}{(2\pi)^{N/2}} \int\limits_{-\infty}^{\infty} \mathrm{d}^N \boldsymbol{r} \exp(-\imath \boldsymbol{k} \cdot \boldsymbol{r}) k^2 \boldsymbol{x}(\boldsymbol{r}, t) = -k^2 \tilde{\boldsymbol{x}}(\boldsymbol{k}, t). \tag{62}$$

Application of the Fourier transform on Eq. (58) yields a linear inhomogeneous dynamical system in time t,

$$\partial_t \tilde{\boldsymbol{X}}_1(\boldsymbol{k}, t) = -k^2 \boldsymbol{\mathcal{D}} \tilde{\boldsymbol{X}}_1(\boldsymbol{k}, t) + \nabla_{\boldsymbol{x}} \boldsymbol{R}(\boldsymbol{X}_0) \tilde{\boldsymbol{X}}_1(\boldsymbol{k}, t) + \tilde{\boldsymbol{f}}(\boldsymbol{X}_0, \boldsymbol{k}, t) \tag{63}$$

$$\tilde{\boldsymbol{X}}_1(\boldsymbol{k}, t_0) = \tilde{\boldsymbol{x}}_1(\boldsymbol{k}) \tag{64}$$

with $\tilde{\boldsymbol{X}}_1(\boldsymbol{k},t)$ and $\tilde{\boldsymbol{f}}(\boldsymbol{X}_0,\boldsymbol{k},t)$ the Fourier transform of $\boldsymbol{X}_1(\boldsymbol{r},t)$ and $\boldsymbol{f}(\boldsymbol{X}_0,\boldsymbol{r},t)$, respectively, and $\tilde{\boldsymbol{x}}_1(\boldsymbol{k})$ the Fourier transform of the initial condition $\boldsymbol{x}_1(\boldsymbol{r})$. To solve Eq. (63) for $\tilde{\boldsymbol{f}} \equiv \boldsymbol{0}$, we proceed analogously to the case of dynamical systems from the last page and determine the eigenvalues $\lambda_i(\boldsymbol{k})$, $i \in \{1,\ldots,n\}$ of the $n \times n$ matrix

$$\boldsymbol{\mathcal{A}} = \nabla_{\boldsymbol{x}} \boldsymbol{R}(\boldsymbol{X}_0) - k^2 \boldsymbol{\mathcal{D}}. \tag{65}$$

The major difference to the case of dynamical systems is that now all eigenvalues become functions of the vector of wave numbers $\boldsymbol{k}$. If the real part of all eigenvalues is strictly negative ,$\Re(\lambda_i(\boldsymbol{k})) < 0$, for all $i \in \{1,\ldots,n\}$ and all wave numbers $\boldsymbol{k} \in \mathbb{R}^N$, the homogeneous stationary solution $\boldsymbol{X}_0$ is stable.

References

[1] Carl M Bender and Steven A Orszag. *Advanced Mathematical Methods for Scientists and Engineers I.* Springer Science & Business Media, 1999.

[2] Hassan K Khalil and JW Grizzle. *Nonlinear systems*, volume 3. Prentice hall New Jersey, 1996.

[3] James D Murray. *Mathematical Biology I: An Introduction.* Springer, New York, USA, 2002.

[4] Ali H Nayfeh. *Perturbation methods.* John Wiley & Sons, 2008.

[5] Ali H Nayfeh. *Introduction to perturbation techniques.* John Wiley & Sons, 2011.