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1. Übungsblatt – TPVI: Theorie des Quantentransportes

Abgabe: Do. 25.10.2019 16:00 Uhr im Tutorium

Bei den schriftlichen Ausarbeitungen werden ausführliche Kommentare zum Vorgehen erwartet. Dafür gibt es auch Punkte! Die Abgabe soll in Zweiergruppen oder Dreiergruppen erfolgen.

Aufgabe 0 (10 Punkte): *Thermal state of a bosonic mode*

Let us consider a harmonic mode with the Hamiltonian $H = \omega a^\dagger a$, where $a^\dagger$ and a are bosonic creation and annihilation operators. Assume the state of the mode corresponds to a thermal state with temperature T, described by $\rho = e^{-\beta H}/Z$, with $Z = \text{Tr}\left\{e^{-\beta H}\right\}$ and $\beta = (k_B T)^{-1}$.

(a) (3) Obtain the average occupation number $\langle n \rangle$. What happens when $T \to 0$ and $T \to \infty$?

(b) (3) Show that the state ρ can be written in terms of $\langle n \rangle$ and the Fock states as

$$\rho = \frac{1}{1 + \langle n \rangle} \sum_m^\infty \left(\frac{\langle n \rangle}{1 + \langle n \rangle} \right)^m |m\rangle\langle m|.$$

(c) (1) What is the probability of finding $n \in \{0, 1, 2, 3 \ldots\}$ bosons?

(d) (3) Calculate the fluctuations in the average occupation number $\langle (\Delta n)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2$.

Aufgabe 1 (6 Punkte): *Thermal state of a fermionic mode*

Let us consider a fermionic mode with Hamiltonian $H = \omega d^\dagger d$, where $d^\dagger$ and d are fermionic creation and annihilation operators. Assume the state of the mode corresponds to a grand canonical thermal state with temperature T and chemical potential μ, described by $\rho = e^{-\beta(H-\mu N)}/Z$, with $N = d^\dagger d$, $Z = \text{Tr}\left\{e^{-\beta(H-\mu N)}\right\}$ and $\beta = (k_B T)^{-1}$.

(a) (3) Obtain the average occupation number $\langle n \rangle$. What happens when $T \to 0$ and $T \to \infty$?

(b) (3) Calculate the fluctuations in the average occupation number $\langle (\Delta n)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2$.

Aufgabe 2 (14 Punkte): *Two coupled systems*

Let us consider two coupled quantum systems. The first one is a two level system described by eigenstates of the Pauli-z matrix $\sigma_z|\pm\rangle = \pm|\pm\rangle$ and the second one is a bosonic mode. The total Hamiltonian is $H = H_0 + H_I$, where H_0 is the free Hamiltonian and H_I their interaction. They are given by

$$H_0 = \omega_0 \sigma_z + \omega a^\dagger a \quad \text{and} \quad H_I = g\left(\sigma_+ a + \sigma_- a^\dagger\right),$$

where σ_z is a Pauli matrix.

(a) (3) Show that $\left[a, (a^\dagger)^n\right] = n(a^\dagger)^{n-1}$ and $\left[a^\dagger, a^n\right] = -na^{n-1}$

(b) (3) What is the ground state of H_0 and what is its energy?

(c) (4) Show that the total number of excitations $N = a^\dagger a + \frac{\sigma_z + 1}{2}$ is a constant. This means the Hilbert space of the problem can be divided into smaller sub-spaces that preserve the number of excitations. What is a possible basis for a subspace with total excitations $n+1$?

(d) (4) The interaction picture Hamiltonian is defined by $\tilde{H}_I(t) = U_0^\dagger(t) H_I U_0(t)$, with $U_0(t) = e^{-iH_0 t}$. Find $\tilde{H}_I(t)$ (Hint: For the case of $2\omega_0 = \omega$, $\tilde{H}_I(t)$ has the same structure as H_I).

Vorlesung:	• Do. 10:00 Uhr – 12:00 Uhr im EW 203. • Fr. 10:00 Uhr – 12:00 Uhr im EW 203.
Übung:	• Do. 16:00 Uhr –18:00 Uhr im EW 733.
Scheinkriterien:	• Mindestens 60% der Übungspunkte. • Regelmäßige und aktive Teilnahme am Tutorium.